Geometric Spreading of $P_n$ and $S_n$ in a Spherical Earth Model

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Abstract  Geometric spreading of $P_n$ and $S_n$ waves in a spherical Earth model is different than that of classical headwaves and is frequency dependent. The behavior cannot be fully represented by a frequency-independent power-law model, as is commonly assumed. The lack of an accurate representation of $P_n$ and $S_n$ geometric spreading in a spherical Earth model impedes our ability to characterize Earth properties including anelasticity. We conduct numerical simulations to quantify $P_n$ and $S_n$ geometric spreading in a spherical Earth model with constant mantle-lid velocities. Based on our simulation results, we present new empirical $P_n$ and $S_n$ geometric-spreading models in the form $G(r, f) = \left[10^{n_1(f)/r_0}\right]_{n_2} log(\text{f} / f_0) + n_3(f)$ and $n_i(f) = n_{i_1}\left[\text{log}(f / f_0)\right]^2 + n_{i_2} \log(f / f_0) + n_{i_3}$, where $i = 1, 2, 3$; $r$ is epicentral distance; $f$ is frequency; $r_0 = 1$ km; and $f_0 = 1$ Hz. We derive values of coefficients $n_{ij}$ by fitting the model to computed $P_n$ and $S_n$ amplitudes for a spherical Earth model having a 40-km-thick crust, generic values of $P$ and $S$ velocities, and a constant-velocity uppermost mantle. We apply the new spreading model to observed data in Eurasia to estimate average $P_n$ attenuation, obtaining more reasonable results compared to using a standard power-law model. Our new $P_n$ and $S_n$ geometric-spreading models provide generally applicable reference behavior for spherical Earth models with constant uppermost-mantle velocities.

Introduction

Seismic phases $P_n$ and $S_n$ are refracted waves that traverse the uppermost mantle and are typically the first $P$ and $S$ arrivals at distances from ~200 to ~1500 km. These phases are commonly characterized as being conical headwaves based on considerations of wave interactions with planar constant-velocity layered structures. However, the propagation of $P_n$ and $S_n$ in the actual spherical Earth is much more complex. In addition to the effects on $P_n$ and $S_n$ propagation of uppermost-mantle velocity complexities such as mantle-lid radial velocity gradients, lateral velocity heterogeneities, and depth variation of the Moho discontinuity, the curvature of both the Moho and the Earth’s surface (the sphericity) alters $P_n$ and $S_n$ amplitudes profoundly from that of a headwave. Textbook treatments of the propagation of conical headwaves in plane crustal-layer-over-half-space models (e.g., Lay and Wallace, 1995; Aki and Richards, 2002) hold for a spherical Earth model only in the unlikely situation of a precisely critical negative velocity gradient below the Moho. Lesser negative velocity gradients, constant velocities, or positive velocity gradients in the mantle will all produce complex frequency-dependent behavior of $P_n$ and $S_n$ that is not the same as that of a conical headwave. At relatively short epicentral distances (less than a few hundred kilometers), the sphericity effects are negligible. But as epicentral distance increases, $P_n$ and $S_n$ behavior is strongly influenced by the sphericity, and one cannot ignore the need for formal mapping between plane-layered structures and spherical velocity structures. The effects of the sphericity must be accounted for when interpreting regional-phase seismic magnitudes, source-type discriminants using $P_n$ and $S_n$, and frequency-dependent measures such as the attenuation of $P_n$ and $S_n$. In this study, we quantify the sphericity effects on $P_n$ and $S_n$ geometric spreading through detailed numerical modeling.

The behavior of $P_n$ and $S_n$ in a spherical Earth model has been studied both theoretically and numerically since the 1960s. Buldrev and Lanin (1965) and Hill (1973) investigated the propagation of $S_n$ and $P_n$ in and around an elastic spherical body by solving the wave equation using asymptotic methods. Červený and Ravindra (1971) used the ray method to characterize the phenomenon of $P_n$ traveling in a plane-layered medium with a constant-velocity gradient in the mantle, which can be related to models with spherical boundaries through the Earth flattening transformation (EFT) (e.g., Chapman, 1973; Müller, 1977). Sereno and Given (1990) conducted numerical simulations of $P_n$ waves in a plane-layered Earth model generated from a spherically symmetric Earth model with the EFT. The conclusions from these studies using different approaches are that (1) the geometric spreading of $P_n$ and $S_n$ in a spherical Earth model differs...
significantly from that of conical headwaves for plane-layered structures; (2) the spreading cannot be accurately represented for all ranges by a simple power-law model; and (3) the spreading is frequency dependent, which is not the case for classic headwaves.

In spite of these conclusions, seismologists commonly use a frequency-independent power-law model to approximate $P_n$ geometric spreading in the real Earth when analyzing observed data (e.g., Sereno et al., 1988; Zhu et al., 1991; Taylor et al., 2002). Occasionally, researchers combine $P_n$ geometric spreading and attenuation into a single distance-decay term in order to avoid the difficulty of isolating $P_n$ geometric-spreading effects (e.g., Chun et al., 1989; Tinker and Wallace, 1997). Because of the apparent incompatibility between the behavior of $P_n$ in the real Earth and its power-law geometric-spreading representation, Taylor et al. (2002) postulated that the use of a frequency-independent power-law spreading model might be the reason for difficulties encountered in fitting the $P_n$ spectra in their study. The common practice of representing $P_n$ geometric spreading in the real Earth with a frequency-independent power-law model may simply be a choice of convenience due to the unavailability of a more accurate, easy-to-use, frequency-dependent $P_n$ geometric-spreading model, even for a simple reference velocity structure. Although there exist theoretical solutions of waves propagating in an elastic body with spherical symmetry (Buldyrev and Lanin, 1965; Hill, 1973), they only describe high-frequency asymptotic wave behavior. These solutions are also too complex to be of general use in practical applications.

Building upon the insight provided by the $P_n$ modeling of Sereno and Given (1990), we perform more detailed and rigorous numerical simulations in this study to establish appropriate $P_n$ and $S_n$ geometric-spreading relationships for the important class of continental 1D spherical Earth models with constant seismic velocities in the uppermost mantle. Prescribing a constant uppermost-mantle velocity is a common default given that detailed and reliable velocity structure and gradients in the mantle lid are either not available or not general enough to represent many parts of the uppermost mantle beneath continents. Based on our modeling results, we first propose a quantitative, frequency-dependent $P_n$ geometric-spreading model. We then explore the implications of this new model on attenuation estimation using observed $P_n$ amplitude data. We also investigate the variation of model parameters, due to variations of crustal thickness, source depth, and crust/mantle velocity contrast, and the resulting variation of attenuation estimates. Finally, we model $S_n$ propagation under the same configuration and propose a similar geometric-spreading model for $S_n$. Even though our spreading models cannot account for all the complexities of $P_n$ and $S_n$ propagation in the real Earth, they serve as better first-order approximations of $P_n$ and $S_n$ geometric spreading in a spherical Earth than the standard power-law model.

Methodology

We conduct most of our simulations using the reflectivity method (Kennett, 1983; Randall, 1994). Results from reflectivity calculations are compared with results calculated with a 2D finite-difference code (Xie and Lay, 1994) and a 2.5D axisymmetric spherical finite-difference code, SHaxi (Igel and Weber, 1995) to confirm that the EFT and layer discretization required by the reflectivity method do not produce numerical artifacts.

The reflectivity method generates complete synthetic seismograms within a specified slowness range for 1D plane-layered velocity models. In order to use the reflectivity method for a spherically symmetric Earth model, we apply the EFT to transform the spherical Earth model to a plane Earth model. Transformations of velocity $v$ and depth $z$ are (Chapman, 1973; Müller, 1977)

$$v_f = \frac{R}{R-z_r} v_r \quad \text{and} \quad z_f = R \ln \left( \frac{R}{R-z_f} \right),$$

where $R$ is the radius of the Earth. Subscript $r$ designates values in the spherical (radially symmetric) model, and subscript $f$ designates values in the plane (flat) model. The density $\rho$ transformation is

$$\rho_f = \left( \frac{R}{R-z_r} \right)^m \rho_r,$$

which is not unique because $m$ can take any value between $-5$ and $1$. For regional body waves, the choice of $m$ is not critical (Müller, 1977). We choose $m = -1$ for $P/SV$ simulations (Müller, 1977) and $m = -5$ for $SH$ simulations (Chapman, 1973). We experimented with different values of $m$ and the results were basically unchanged. Finally the transformation of amplitudes calculated from plane-model simulations back to corresponding amplitudes in the spherical model is

$$A_f = \left( \frac{\Delta}{\sin \Delta} \right)^{1/2} \left( \frac{R}{R-Z_r} \right)^{(m+5)/2} A_r,$$

where $Z_r$ is the depth of the source in the spherical model and $\Delta$ is epicentral distance in radians.

We approximate the velocity gradient resulting from the EFT (equations 1 and 2) with homogeneous layers in the plane Earth model, as is required by the reflectivity method. The thickness of these layers affects the accuracy of the approximation, with thinner layers yielding more accurate results. We set the thickness of these layers to be about 0.4 of the minimum wavelength of the waves to be modeled, which appears to be more than adequate (Chapman and Orcutt, 1985). Further reducing the ratio (e.g., from 0.4 to 0.2 of the minimum wavelength) does not alter the results appreciably. The total thickness of the gradient zone is set to be more than 100 km larger than the maximum penetration depth of
the direct wave in a homogeneous spherical model recorded at the longest epicentral distance considered. This thickness guarantees that no $P_n$ or $S_n$ waves observed within the distance range of interest are affected by the lower boundary of the gradient zone. Below the gradient zone, the velocity is constant.

We use the same generic spherical Earth model considered by Sereno and Given (1990) as the reference model for our simulations and use the synthetics from the simulation to derive parameters of the $P_n$ and $S_n$ geometric spreading that we develop. The reference Earth model consists of a 40-km-thick outer layer, representative of an average continental crust, with a constant-velocity mantle underneath (Fig. 1). The model has no anelastic attenuation. The simplicity of this model allows us to isolate the effects of the sphericity on $P_n$ and $S_n$ geometric spreading. We use an isotropic point source for $P_n$ simulations. The source for $S_n$($SH$) simulations is a vertical strike-slip source, and the source for $S_n$($SV$) simulations is a vertical dip-slip source. For all source types in our main calculations, a delta function is used as the source time function; source depth is 15 km, and source strength is $10^{15}$ N m. Three-component synthetic ground displacements are computed at 33 locations distributed log-evenly along a linear profile from 200 (1.8°) to 2500 km (22.5°). The Nyquist frequency of the seismograms is 20 Hz.

We cut $P_n$ and $S_n$ portions of the synthetic seismograms using fixed-velocity windows. The velocities that we use to define the widths of the $P_n$ windows are 7.6 and 8.2 km/sec, and those for $S_n$ windows are 4.0 and 4.7 km/sec (Hartse, et al., 1997). The windows are centered at the peaks of the phases. We also tested a fixed-window-width method, and the results remained essentially the same. We window $P_n$ and $S_n$($SV$) from vertical-component seismograms and $S_n$($SH$) from transverse-component seismograms. After $P_n$ and $S_n$ seismograms are windowed, we taper the seismograms with small tapers (between 2% to 20% depending on the length of the signal relative to the window length) and Fourier transform the seismograms to obtain the amplitude spectra. We make spectral-amplitude measurements at 100 frequencies log-evenly distributed between 0.75 and 13 Hz. Amplitude at each frequency $f_i$ is calculated by taking the average of the amplitudes between frequencies $f_i/\sqrt{2}$ and $\sqrt{2}f_i$ (Bowman and Kennett, 1991; Hartse et al., 1997).

To accurately assess the geometric spreading of seismic phases, the propagation medium used for the simulation should have no attenuation. However, in order to avoid a computational singularity, the reflectivity method requires a nonzero amount of attenuation for the medium. We take an asymptotic approach similar to that used by Yang (2002) to derive $P_n$ and $S_n$ amplitudes for an elastic model without attenuation from amplitudes calculated for a group of anelastic models. We first make 20 calculations for models that have attenuation quality factor $Q$ log-linearly increasing from 10,000 to 100,000. For each calculation, a single $Q$ is used for both $P$ and $S$ waves and for all parts of the model. Amplitudes at each frequency and each epicentral distance from these calculations are then fit by a quadratic polynomial as a function of $1/Q$. The limit of the polynomial as $Q$ approaches infinity is taken as the amplitude at that frequency and distance for the elastic model.

**$P_n$ Modeling Results**

Figure 2 plots the vertical synthetic $P_n$ seismograms from the reference-model simulation at selected epicentral distances. $Q$ used in this simulation is 100,000. The seismograms are low-pass filtered below 10 Hz to suppress numerical noise near the Nyquist frequency. The figure reveals several interesting characteristics of $P_n$ traveling in a

**Figure 1.** Reference Earth model used for $P_n$ and $S_n$ simulations and the development of new $P_n$ and $S_n$ geometric-spreading models. Quality factor $Q$ is infinite throughout the model.

**Figure 2.** Synthetic $P_n$ seismograms from reference-model calculations. The seismograms are filtered below 10 Hz. Travel time is reduced by 8.2 km/sec. Only every other trace calculated is plotted to enhance clarity. $r$ is the epicentral distance in kilometers.
spherical Earth model with constant mantle velocities. Due to the sphericity, the apparent $P_n$ velocity is not constant but varies with epicentral distance. As is predicted by theory (e.g., Červený and Ravindra, 1971), the pulse shape of $P_n$ evolves from that of the impulse source at distances close to the critical distance (about 0.8° for the reference model and a 15-km-deep source) to the shape of a far-field body wave, which is the time derivative of the source pulse, at farther distances. The amplitude of the phase changes in a complex manner, first decreasing and then increasing, within this distance range. At about 10° to 12°, the first pulse separates from the rest of the $P_n$ wave packet, and somewhere between 16° and 19°, a second pulse separates.

For high-frequency $P_n$ at distances away from the critical distance, Červený and Ravindra (1971) offer a detailed description of the signal behavior from ray theory. Although Červený and Ravindra (1971) describe the phenomenon for a plane-layered Earth model with a positive and constant-velocity gradient in the mantle, their description and conclusions are applicable to the spherical Earth model situation as well because the spherical model can be mapped, through the EFT, into a plane-layered model with an approximately constant-velocity gradient in the uppermost mantle. Following Červený and Ravindra (1971), the $P_n$ phase at distances between about 5° and 10° in Figure 2 can be thought of as the superposition of individual waves reflected $n$ times ($n = 0, 1, 2, \ldots$) from the underside of the Moho. The superposed wave is termed interference headwaves by Červený and Ravindra (1971) and is likened to the “whispering gallery” phenomenon by Menke and Richards (1980). As distance increases, individual components of the interference headwave start to separate from the wave packet due to their increasingly shorter path lengths compared with path lengths of the remaining waves in the wave packet. The first wave to separate is the wave that has no reflection at the Moho (the direct or diving wave). This is evidenced as the separation of the first pulse in Figure 2. The second separated pulse in the figure is the wave that is reflected once from the Moho. From ray theory, the epicentral distance at which the $k$-time reflected wave separates from the interference headwave packet is (Červený and Ravindra, 1971, equation 6.4)

$$r_k = (2H - d) \frac{v_c}{\sqrt{v_m^2 - v_c^2}} + \left[ \frac{32 v_m^2 T (1 + k)^2 (1 + \frac{k}{2})^2}{g^2 (1 + \frac{3}{2} k)} \right]^{1/3},$$  \hspace{1cm} (4)

where $H$ is the crustal thickness; $d$ is the source depth; $v_c$ is the $P$-wave velocity of the crust; $v_m$ is the $P$-wave velocity at the top of the mantle; $T$ is the pulse width of the wave, and $g$ is the velocity gradient ($dv/dz$) at the top of the mantle resulting from the EFT. From equation (4) and for the reference model, a wave with a pulse width of about 0.1 sec (10 Hz) that undergoes no reflection at the Moho will separate at about 9.8°, and the wave that has one reflection will separate at about 16.6°. These predictions are consistent with the synthetics in Figure 2.

Figure 3 shows the amplitude spectra of synthetic $P_n$ at the same epicentral distances as those in Figure 2 from the $Q = 100,000$ computation. The figure illustrates the evolution of the $P_n$ spectrum from being proportional to the source spectrum to being proportional to the time derivative of the source spectrum.

As was described in the last section, we use an asymptotic method to derive spectral amplitudes for an elastic model from amplitudes obtained using anelastic simulations. Figure 4 gives an illustration of the method. Plotted in the figure are $P_n$ amplitudes at different frequencies measured from calculations using different quality factors. The amplitudes are computed at 22.5° epicentral distance and are normalized by the maximum amplitude in the figure. Quadratic polynomial fits of the amplitudes are plotted as solid lines. The polynomial fits are almost perfect, indicating that our approach is appropriate. Amplitudes at other distances are fit as well as those shown in Figure 4.

To visualize the $P_n$ amplitude decay in a spherical Earth model, we plot 10-Hz $P_n$ amplitudes for the reference model in Figure 5. We extend the epicentral-distance range to between 135 (1.2°) and 8000 km (71.9°) for this particular simulation in order to better depict the evolution of $P_n$ waves. Amplitudes at distances beyond about 20° are measured from the direct wave that has been completely separated from the interference headwaves. The amplitudes are corrected for the free-surface effect (the Appendix), which is only important at teleseismic distances. Also plotted in the figure are the amplitude decay of a conical headwave in a plane one-layer-over-half-space model (Aki and Richards, 2002; equa-

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**Figure 3.** Amplitude spectra of $P_n$ at the same distances as those in Figure 2. Some of the distances are marked on the left of the corresponding spectra.
and the amplitude decay of an infinite-frequency direct wave in a spherical Earth model from ray tracing. At distances close to the critical distance, $P_n$ geometric spreading behaves like that of a conical headwave. As distance increases, $P_n$ spreading starts to deviate from that of the headwave, and at about 5°, $P_n$ amplitudes begin to increase. As was mentioned before, 10-Hz direct-wave energy would separate from the rest of the interference headwave at about 10°. It seems from the figure that this separation is manifested in a change in the smoothness of the $P_n$ amplitude variation followed by a reduced rate of amplitude increase. In the range beyond the critical distance and before the direct-wave separation, $P_n$ evolves from a wave similar to a conical headwave to the interference headwave, which is a superposition of multiple waves reflected from the Moho. As the epicentral distance approaches teleseismic distances, the direct-wave spreading approaches that of the infinite-frequency wave from ray tracing results, as is expected. The direct wave dominates the whole $P_n$ wave packet at long distances. We do not see a significant difference between spectral amplitudes obtained by windowing the whole $P_n$ wave packet and those obtained by windowing just the direct wave after its separation from the packet. This is consistent with theoretical predictions (Červený and Ravindra, 1971).

$P_n$ geometric spreading in a spherical Earth model is not only different from that of a head wave, as is shown in Figure 5, but also is frequency dependent. Figure 6 shows the $P_n$ amplitude-variation surface as a function of distance and frequency for the reference model. The strong frequency dependence of the amplitudes is apparent. Amplitudes at higher frequencies are affected more by the sphericity than are lower-frequency amplitudes. The separation distance of the direct wave from the interference headwaves becomes shorter as frequency becomes higher (equation 4).

A New $P_n$ Geometric-Spreading Representation for a Spherical Earth Model

Figures 5 and 6 illustrate that a frequency-independent power-law model cannot accurately represent $P_n$ geometric spreading in a spherical Earth model. Such a representation would plot as a straight line in Figure 5, which is clearly inappropriate for modeling $P_n$ geometric spreading over a wide distance range. In addition, a power-law model with a constant exponent does not take into account the frequency dependence of $P_n$ spreading shown in Figure 6. Based on the $P_n$ amplitude-decay behavior shown in Figures 5 and 6, we propose a new empirical $P_n$ geometric-spreading model that fits the synthetic data much better and that also results in more reasonable anelastic-attenuation estimates from observed data, as we will discuss in more detail in the next section.

The amplitude spectrum of $P_n$ can be parameterized as

$$A(r, \theta, f) = K(f)M_0R(\theta)G(r, f) \exp\left(-\frac{\pi f}{Q(f)\nu}r\right)S(f),$$

where

- $K(f)$ is the frequency-dependent amplitude factor,
- $M_0$ is the moment of the earthquake,
- $R(\theta)$ is the spherical harmonic function,
- $G(r, f)$ is the frequency-dependent amplitude factor,
- $S(f)$ is the frequency-dependent attenuation factor,
- $Q(f)$ is the quality factor, and
- $\nu$ is the shear wave velocity.

Equation (5) provides a comprehensive model for $P_n$ geometric spreading in a spherical Earth model.
In equation (5), $K$ is a frequency-dependent scaling factor; $M_0$ is the source moment; $R$ is the source radiation pattern; $Q$ is the $P_n$ quality factor; $v$ is the $P_n$ velocity; $S$ is the receiver site response; $r$ is the epicentral distance; $\theta$ is the azimuth angle, and $f$ is frequency. $r_0$ and $f_0$ are included in equations (6) and (7) in order for the new model to have the same dimension as standard power-law models (e.g., Street et al., 1975; Sereno et al., 1988). The main differences between the new geometric-spreading model (equations 6 and 7) and the standard frequency-independent power-law model are the addition of the first term in the exponent and the frequency dependence of parameters $n_i$. In the logarithm domain, the new model is a quadratic function of log-distance, whereas the power-law model is linear. The reason for choosing a log-quadratic function is to keep the parameterization as simple as possible while providing a good fit to the synthetics. The adoption of a quadratic functional form for $n_i$ (equation 7) is based on the behavior of $n_i$ versus the frequency obtained by fitting equation (6) to synthetic $P_n$ amplitudes at individual frequencies.

If we take the common logarithm of equation (6), substitute equation (7) into the result, and let $r_0$ and $f_0$ equal one, we obtain

$$
\log[G(r, f)] = n_{i1} \log^2 f + n_{i2} \log f + n_{i3}
$$

(8)

where $r$ is in kilometers and $f$ is in hertz. To derive coefficients $n_{ij}$, we fit equation (8) to synthetic $P_n$ amplitudes shown in Figure 6 in a least-squares sense. $P_n$ amplitudes are corrected for $M_0$ used in the simulation ($10^{15}$ Nm) and $K$ before the fitting. Because the source that we use in the simulation has a flat spectrum, $K$ is frequency independent. We use $K = (4\pi\rho v^2)^{-1}$ (Denny and Johnson, 1991), where $\rho$ is the density and $v$ is the $P$-wave velocity of the source region. Source radiation and site response are unity. We use $P_n$ amplitudes at epicentral distances beyond 300 km ($2.7^\circ$) and before the start of the direct-wave separation to fit the model. We use 300 km as the lower distance limit because reliable $P_n$ observations are typically made at some distances beyond the $P_g$ crossover distance (~200 km). The choice of 300 km is also to avoid possible long-period numeric-noise contamination at short distances, as is indicated in Figure 6. The upper distance limits are based on the observation that within these distances $P_n$ is the result of the interference of all of its components including the direct wave. At larger distances, the direct wave separates from

Figure 5. 10-Hz synthetic $P_n$ amplitude decay in a spherical Earth model with constant mantle velocities. The solid line depicts the theoretical amplitude decay of a conical headwave in a plane one-layer-over-half-space Earth model. The dashed line is the amplitude decay of the infinite-frequency direct wave in a spherical homogeneous Earth model from ray-tracing calculations. Because we are interested in comparing only the decay of the amplitude curves, they are shifted in the vertical direction arbitrarily (normalized) so that they overlap.

with the new geometric-spreading model expressed as

$$
G(r, f) = \frac{10^{QG(f)} r_0}{r} (r_0/r)^{n_i0 + n_if} \quad (r_0 = 1 \text{ km})
$$

(6)

and

$$
n_i(f) = n_{i1} \left[ \log \frac{f}{f_0} \right]^2 + n_{i2} \log \frac{f}{f_0} + n_{i3}
$$

(7)

\begin{align*}
(i = 1, 2, 3; f_0 = 1 \text{ Hz}).
\end{align*}
the rest of the wave packet, and the characteristics of $P_n$ become different. The $P_n$ amplitude decay within the defined distance range also has a smooth pattern and thus is easier to fit by a simple mathematical model. The upper distance limits vary from 7.3° to 17.3° for the frequency range between 13 and 0.75 Hz. Within the specified distance limits, the new spreading model is applicable. Because at about 15° $P_n$ in the real Earth is overtaken by upper-mantle triplications resulting from reflections and refractions at 410- and 660-km discontinuities (e.g., Lay and Wallace, 1995) and is no longer the first arrival, $P_n$ is usually used within the distance range where the new spreading model is valid for frequencies below about 2 Hz. For higher frequencies, the range of applicability of the new model is shorter, but observationally high-frequency signals are generally only detectable above the noise level at shorter distances. Coefficients $n_{ij}$ ($i = 1, 2, 3; j = 1, 2, 3$) from the fitting are listed in Table 1. The inclusion of $r_0$ and $f_0$ in the model also guarantees that, even though the values of the coefficients are derived using equation (8) with $r$ in kilometers and $f$ in hertz, they are valid for $r$ and $f$ in any units as long as $r_0$ and $f_0$ are converted accordingly.

Figure 7 compares the new geometric-spreading model and a power-law model with synthetic $P_n$ amplitudes. The power-law model has an exponent of $-1.3$ (Sereno et al., 1988). The difference between the new spreading model and synthetic $P_n$ amplitudes is almost indistinguishable. On the other hand, the power-law model deviates from synthetic $P_n$ amplitudes significantly.

Application to Observed Data

The key value of any mathematical model of the physical world is for the model to be able to provide physically reasonable descriptions of observed data. To test the validity and usefulness of the new $P_n$ geometric-spreading model, we correct a set of observed $P_n$ spectral amplitudes for geometric spreading with the new model and estimate the average medium attenuation. We then compare the results with those published in the literature.

We represent observed $P_n$ amplitudes by equation (5). For the purpose of testing the new $P_n$ geometric-spreading model, we simplify equation (5) by assuming that site response is unity for all stations and source radiation patterns can be ignored. We presume that errors introduced by these simplifications are random and should not affect average-attenuation estimates systematically. With known or estimated source moments, an assumed scaling factor $K$, and a $P_n$ geometric-spreading model, we can estimate the average-attenuation quality factor at each frequency by least-squares fitting the logarithm of source and geometric-spreading corrected spectral amplitudes as a function of epicentral distance.

The observed $P_n$ amplitudes are measured on vertical-component ground-displacement data recorded by stations in and around China and in southern Europe for events in the same region. The same windowing method as the method we employ to measure the synthetic $P_n$ amplitudes is used. Analyst picks reported in global catalogs (from the International Seismological Centre, the U.S. Geological Survey, the International Data Center of the Preparatory Commission for the Comprehensive Nuclear-Test-Ban Treaty Organization, etc.) are used to center the $P_n$ windows. Randall et al. (2006) give a more detailed description of the amplitude dataset. We derive source moments from body-wave magnitudes ($m_b$) reported in the catalogs using the relationship developed by Geller (1976). We use amplitudes only from events with $m_b$ equal to or smaller than 6 to avoid magnitude saturation. We use a simplified version of the scaling factor $K$ expressed as $K(f) = [4\pi \rho v^2 [1 + (f/f_c)^2]]^{-1}$ using crustal P-wave velocity and the density of the reference Earth model as $v$ and $\rho$. The source corner frequency $f_c$ is calculated from the source moment using the relationship $\log M_0 = 17.08 - 3.24 \log f_c$ derived by Xie and Patton (1999) from $P_n$ amplitude data recorded in central Asia. For comparison, we use both the new $P_n$ geometric-spreading model and the power-law model with two different exponents, $-1.1$ (Walter and Taylor, 2002) and $-1.3$, in the attenuation estimation. When the new spreading model is used, we limit the epicentral distances of the amplitudes used in the estimation to within the distance range where the model is valid. For power-law model corrections, we use amplitudes between 300 (2.7°) and 1668 km (15°). Attenuation is estimated at 0.75, 1.0, 2.0, 4.0, and 6.0 Hz. Figure 8 plots the 1-Hz $P_n$ amplitudes after source and geometric-spreading corrections. The new spreading model is used in the correction. Although the amplitudes show a large scatter, a linear decay trend due to realistic values of medium attenuation is discernible.

Table 2 lists estimated average quality factors using different geometric-spreading corrections from observed $P_n$ amplitudes. Using the power-law spreading model results in some negative values at low frequencies. At high frequencies, the power-law model yields estimates that range from over 1000 to over 5000. On the other hand, $Q$ estimates using the new spreading model are positive and below 700 at all frequencies. These values can be compared with published $P$-wave quality factors in the mantle lid, as we discuss in the following paragraphs.

<table>
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<tr>
<th>Table 1</th>
<th>Coefficients of the New $P_n$ Geometric-Spreading Model</th>
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<tbody>
<tr>
<td>$a_{11}$</td>
<td>$a_{12}$</td>
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<td>-0.217</td>
<td>1.79</td>
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</tbody>
</table>
Using theory, observed body-wave spectra, and waveform modeling, Lundquist and Cormier (1980) derive generic absorption-band $\mathcal{P}$-wave $\mathcal{Q}$ models for the mantle. The $\mathcal{Q}$ values of these models range from about 100 to 500 for frequencies between 0.7 and 6.0 Hz in the depth range of 45 to 200 km. In their article, Lundquist and Cormier (1980) also cite results of some other $\mathcal{Q}$ studies that use free-oscillation data, long-period surface waves, and high-frequency (1–5 Hz) body waves. The frequency-independent $\mathcal{P}$-wave $\mathcal{Q}$ models from these studies have values from about 100 to 250 for depths between 50 and 150 km. Der et al. (1986) construct a $\mathcal{P}$-wave $\mathcal{Q}$ model for the Eurasian Shield using a large set of teleseismic body waves. Their model has values between about 350 and 900 for frequencies between 0.3 and 10 Hz at depths between 100 and 200 km. Above 100 km, $\mathcal{Q}$ values increase to between 600 and 1500 for the same frequency range.

More recently, some studies make direct $P_n\mathcal{Q}$ estimations. Sereno et al. (1988) and Sereno (1990) obtain $P_n\mathcal{Q}$ models for Scandinavia and eastern Kazakhstan, respectively, by inverting broadband $P_n$ spectra. The 0.75- to 6-Hz $P_n\mathcal{Q}$ values that they estimated are between 283 and 768 for Scandinavia and between 260 and 735 for eastern Kazakhstan. Although Sereno et al. (1988) and Sereno (1990) assume a power-law $P_n$ geometric-spreading model with an exponent of $10^{-1.1}$, their $P_n\mathcal{Q}$ estimates are more in line with the average $P_n\mathcal{Q}$ estimates that we obtain using the new $P_n$ spreading model than with those from power-law model corrections (Table 2). A possible explanation for this observation is that the majority of their data are recorded within 1000-km epicentral distance. At short distances, the power-law spreading model has a gentler slope than the new spreading model does (Fig. 7) and therefore would yield smaller $\mathcal{Q}$ estimates from short-distance data. However, for a broader distance range such as the distance range that our dataset covers, the power-law model yields larger, sometimes negative, $\mathcal{Q}$ estimates because of the steeper slope of the model at long distances (Fig. 7). The implication is that if a power-law $P_n$ spreading model with a specific exponent is used, it will be applicable only as an approximation in a limited distance range, and models with different exponents are needed for different distance ranges. Our parameterization remedies this failing.

Comparing $\mathcal{Q}$ values in Table 2 with those published in the literature, we conclude that the new $P_n$ geometric-spreading model yields $\mathcal{Q}$ estimates that are generally consistent with published results over the broad region of Eurasia. The $\mathcal{Q}$ estimates with power-law model corrections, on the other hand, have values that are either negative or seem to be too large. It should be noted that $\mathcal{Q}$ values estimated in this fashion represent only the average $P_n$ attenua-
Geometric Spreading of $P_n$ and $S_n$ in a Spherical Earth Model

Variation of Model Parameters Due to Variations of Earth-Model Properties

The new $P_n$ geometric-spreading model is derived by fitting synthetic amplitude data calculated for the reference Earth model. The synthetic $P_n$ amplitudes, and thus the spreading-model parameters, are dependent on the Earth model used in the simulation, as is the case for most general geometric-spreading representations. In order to gauge the dependence of the spreading-model parameters on the variation of 1D Earth-model properties, we conduct additional simulations with different crustal thickness, source depth, and crust/mantle velocity contrast configurations.

We test the effects of crustal thickness on spreading-model parameters by simulating $P_n$ propagation in Earth models with crustal thickness changing from 20 to 70 km, a range of typical of continental crusts (Lay and Wallace, 1995). To test the effects of source depth on the spreading-model parameters, we put the source at depths from 10 to 39 km in a series of simulations. We also vary the crust/mantle velocity contrast to test its effects on the spreading-model parameters. We change the mantle $P$-wave velocity from 7.6 to 8.6 km/sec with crustal $P$-wave velocity fixed at 6.5 km/sec. Mantle $S$-wave velocity and density are adjusted in a similar way. Changing crustal velocities and densities to achieve the same contrast range yields similar results.

Figure 9 shows variations of coefficients $n_{ij}$ relative to reference-model results due to variations of crustal thickness, source depth, and crust/mantle velocity contrast. Most of the large variations of the coefficients are caused by crustal-thickness variation. The effect of the source-depth change is moderate. Crust/mantle velocity contrast has the least effect on the coefficients. Among the coefficients, $n_{ij}$ are the most sensitive, though they are least important in contributing to the spreading model due to their relatively small absolute values (Table 1). Although the coefficients vary considerably due to Earth-model property variations, the overall behavior of the spreading model $G(r, f)$ remains stable. The maximum variation of $G(r, f)$ resulting from coefficient variations shown in Figure 9 is less than 4% for a crustal thickness less than 60 km and less than 10% for a thick crust. Figure 10 plots the variation of $Q$ estimated from observed $P_n$ data due to variation of the coefficients. The plot again shows the relative importance of different Earth model properties in affecting the spreading model. The majority of $Q$ variations are within 10% to 20%. The only large $Q$ variations occur for very thick crust and at long periods.

$S_n$ Simulations

In addition to simulating $P_n$ propagation in a spherical Earth model, we also simulate $S_n$ propagation in the same reference model. Except for different source types and different slowness integration limits, other modeling parameters in the $S_n$ simulation are kept the same as those used in the $P_n$ simulation. Figure 11 plots the $S_n$($SH$) seismograms low-pass filtered below 10 Hz. Compared with Figure 2, the behavior of $S_n$ in a spherical Earth model is very similar to the behavior of $P_n$ waves. The only difference is that the separation of individual waves from the interference wave packet occurs at shorter distances for $S_n$. This difference can be predicted using equation (4), although the equation was originally derived only for $P_n$ waves. $S_n$($SH$) spectral amplitudes also form a surface with a shape similar to that of the $P_n$ amplitude surface shown in Figure 6.

Because of the similarities between $P_n$ and $S_n$ propagation in a spherical Earth model with constant mantle velocities, we propose an $S_n$ geometric-spreading model that has the same functional form as that of the new $P_n$ spreading model (equations 6 and 7). We derive the values of the coefficients $n_{ij}$ by fitting synthetic $S_n$($SH$) amplitudes. $S_n$($SV$) amplitudes are severely contaminated by $P$-wave energy up to 1000 km and thus are not suitable for fitting. Beyond 1000 km, $S_n$($SV$) and $S_n$($SH$) amplitudes decay similarly. This suggests that the $S_n$ spreading model developed by fitting $S_n$($SH$) amplitudes is also suitable for describing $S_n$($SV$) geometric spreading.

We correct synthetic $S_n$($SH$) amplitudes for the source moment, the source radiation pattern using takeoff angles calculated with the method described in the Appendix, and the scaling factor $K$ before the fitting. We use $K = (4\pi \rho v^3)^{-1}$, but now $v$ is the $S$-wave velocity of the crust. We again set the lower-distance limit to 300 km (2.7°) for amplitudes used in the fitting. The upper distance limits are from 6.2° to 14.4° for frequencies from 13 Hz to 0.75 Hz. The limits set the distance range within which the $S_n$ spreading model is valid. Table 3 lists the coefficients $n_{ij}$ for the $S_n$ geometric-spreading model from the fitting.

Discussion

The continental uppermost mantle, where the main portions of the $P_n$ and $S_n$ travel paths are located, is one of the most complex regions in the Earth. The geometric and elastic complexities of the uppermost mantle that affect $P_n$ and $S_n$ geometric spreading include the variation of Moho depth, the mantle-lid velocity gradient, and lateral velocity heterogeneities. Some of these complexities may have comparable or larger effects on $P_n$ and $S_n$ geometric spreading.
than the Earth’s sphericity does in some regions of the world. For example, Xie and Patton (1999) attribute the large $P_n$ amplitude variation across a small-aperture seismic array in central Asia to the varying Moho depth in the region. Zhao (1993) obtain a mantle-lid $P$-wave velocity gradient of $8.0 \times 10^{-4}$ sec$^{-1}$ for the Basin and Range province in the United States. Using the same method, Zhao and Xie (1993) estimate a $P$-wave mantle-lid velocity gradient of $3.1 \times 10^{-3}$ sec$^{-1}$ for the Tibetan Plateau region in China. The average uppermost-mantle $P$-wave velocity gradient for northern Eurasia that Morozova et al. (1999) derive from the Russian Deep Seismic Sounding data is on the order of $4 \times 10^{-3}$ sec$^{-1}$. For comparison, the reference-model $P$-wave velocity gradient resulting from the EFT, which causes the geometric spreading of $P_n$ and $S_n$ to deviate significantly from that of the conical head waves, is $1.3 \times 10^{-3}$ sec$^{-1}$ at the top of the mantle.

Whereas strong positive mantle-lid velocity gradients constrained from observed data are reported in some regions of the world, there are also studies with different results and conclusions. Hill (1971) infers from seismic data and laboratory measurements that a negative crustal and upper-mantle velocity gradient is likely to exist in high heat-flow regions like the Basin and Range province. Tittelgemeyer et al. (2000) argue that, from a petrological and petrophysical point of view, a widespread positive velocity gradient in the upper

Figure 9. Variations of coefficients $n_{ij}$ of the $P_n$ spreading model due to variations of crustal thickness, source depth, and crust/mantle velocity contrast. Variations are presented as coefficients for different Earth-model properties normalized by reference-model coefficients in Table 1.
mantle is not expected and that the observation of teleseismic $P_n$ in the Russian Deep Seismic Sounding data can be explained by the existence of lateral velocity heterogeneity in the uppermost mantle. This model is in contrast with the model advocated by Morozov et al. (1998) and Nielsen and Thybo (2003) that the presence of the strong positive upper-mantle velocity gradient and lower-crust velocity heterogeneity is the reason for the observed teleseismic $P_n$.

Even though unique and reliable determination of the detailed uppermost-mantle velocity structure remains challenging, it is evident that the continental uppermost mantle generally has complexities that affect, sometimes strongly, the $P_n$ and $S_n$ propagation including their geometric spreading. The geometric-spreading models that we propose serve only as first-order approximations that account for the effect of the Earth’s sphericity for the simple reference model with constant mantle-lid velocities. Nevertheless, our geometric-spreading models are more appropriate for representing $P_n$ and $S_n$ behavior in the real Earth than the standard power-law model. This conclusion is supported by synthetic simulations and by our successful application of the model on observed $P_n$ data spanning wide distance ranges in Eurasia to yield reasonable attenuation estimates.

The new $P_n$ and $S_n$ geometric-spreading models are useful in common situations where only simple velocity models with uppermost-mantle structure represented as constant-velocity half-space are available. If the mantle-lid velocity gradient is well resolved in a given region, simulations for that gradient can be performed to obtain appropriate geometric-spreading corrections. Because the effect of sphericity is equivalent to the effect of a positive velocity gradient in a plane-layered model, we anticipate that the functional form of our geometric-spreading models will remain the same for Earth models in which an effective (physical plus effects of the sphericity) positive velocity gradient exists. Only the coefficients will differ.

Quantifying the effects of specific mantle-lid velocity gradients, Moho irregularity and lateral velocity heterogeneity in the uppermost mantle on $P_n$ and $S_n$ geometric spreading through 2D and 3D numerical modeling will be the subject of a separate study. It is reasonable to assume that, except for velocity gradients, these effects contribute primarily to scatter around the fundamental behavior of our geometric-spreading models.

**Conclusions**

We perform detailed numerical modeling to characterize $P_n$ and $S_n$ propagation in a spherical Earth model with con-
stant mantle velocities. The results show that $P_n$ and $S_n$ behave in a complex manner in a spherical Earth model, which is consistent with theory predictions. The geometric spreading of $P_n$ and $S_n$ evolves from that of a conical head-wave in a plane-layered Earth model at distances close to the critical distance to that of a direct or diving wave at teleseismic distances. More complexities are introduced by the interference of waves reflected at the underside of Moho and the separation of individual components from the interference wave packet, especially the separation of the direct wave. The complex patterns of $P_n$ and $S_n$ geometric spreading are also frequency dependent with higher frequencies being more affected by the sphericity.

Based on the modeling results, we propose empirical frequency-dependent $P_n$ and $S_n$ geometric-spreading models for a category of spherical Earth models with constant mantle velocities. The new spreading models accurately capture synthetic $P_n$ and $S_n$ amplitude behavior. The $P_n$ spreading model also results in reasonable average-attenuation estimates when applied to observed $P_n$ amplitude data for a broad region of Eurasia.

Variations of Earth-model properties such as crustal thickness, source depth, and crust/mantle velocity contrast within reasonable ranges cause considerable variations of spreading-model parameters. However, the overall behavior of the spreading model is insensitive to variations of these Earth-model properties. Except for long-period waves traveling through very thick crusts, variations of estimated $Q$ from observed data resulting from variations of model parameters are within 20%.

If the velocity structure in the uppermost mantle can be accurately determined for a given region, specific geometric-spreading relations should be predicted by numerical simulations. Otherwise, our new $P_n$ and $S_n$ geometric-spreading models provide reference behavior of $P_n$ and $S_n$ spreading for the common class of spherical Earth models with constant uppermost-mantle velocities. The use of these models should result in a smaller error in different applications compared with using power-law models.

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References


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Table 3

Coefficients of the New $S_n(SH)$ Geometric-Spreading Model

<table>
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<tr>
<th>$n_{11}$</th>
<th>$n_{12}$</th>
<th>$n_{13}$</th>
<th>$n_{21}$</th>
<th>$n_{22}$</th>
<th>$n_{23}$</th>
<th>$n_{31}$</th>
<th>$n_{32}$</th>
<th>$n_{33}$</th>
</tr>
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<tbody>
<tr>
<td>−0.347</td>
<td>2.16</td>
<td>3.54</td>
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<td>10.1</td>
<td>20.4</td>
<td>−4.38</td>
<td>11.7</td>
<td>23.1</td>
</tr>
</tbody>
</table>


Appendix A

Incident (Takeoff)-Angle Calculation and Free-Surface Effect Correction

We calculate amplitudes of incoming Pn waves by removing free-surface effects from vertical ground-displace-ment amplitudes recorded at the Earth’s surface. We use the reflection coefficients for plane interfaces (e.g., Lay and Wallace, 1995, Table 3.1) to approximate reflection coefficients for the curved interface. Because we use the corrected amplitudes only for qualitative comparisons (Fig. 5), this approximation is accurate enough.

Different components of the Pn wave packet have different incident angles for a given distance. We use the incident angle of the direct-wave component as an approximation of the incident angle of the whole Pn wave packet at short distances. Because the direct wave is the main component of Pn and the incident angle varies significantly only at long epicentral distances where we measure Pn amplitudes from the direct wave, this approximation is justified. Ignoring the velocity gradient in the single-layer crust resulting from the EFT, equation 6.8 of Červený and Ravindra (1971) states that the incident angle θ of the direct-wave component of Pn is related to epicentral distance r as

\[ r = (2H - d) \tan \theta + \frac{2v_m \sqrt{v_c^2 - v_m^2 \sin^2 \theta}}{v_m \sin \theta}. \]  

(A1)

The symbols in equation (A1) have the same meanings as those in equation (4) in the main text. For a certain epicentral distance r, we can find the corresponding Pn incident angle at the Earth’s free-surface from equation (A1). Tests have shown that taking the velocity gradient in the crust from the EFT, which affects the first term on the right-hand side of equation (A1), into consideration in calculating the incident angle yields essentially the same result.

The takeoff angle of Pn at the source is the same as its incident angle at the free-surface. We assume that equation (A1) can also be used to calculate the Sn takeoff angle.

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