Green’s function and T-matrix reconstruction using surface data for direct nonlinear inversion
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Summary

We propose to apply the Bojarski equation for the Green’s function and T-matrix reconstruction using surface data, in which only the background Green’s function is assumed to be known. With this reconstruction, no weak-scattering assumption is needed, and direct nonlinear inversion can be applied to the reconstructed new data sets. The key problem is the removal of anti-causal scattered waves. We discussed several approaches to solve this problem.

Introduction

Green’s function retrieval has been a powerful method to reconstruct Green’s function for imaging, tomography and inversion (see, Schuster, 2009; Wapenaar et al., 2008). However, the method needs to put receivers inside the medium. There are new proposed methods of wavefield reconstruction using surface data only, but some extra information such as the first arrival data, is needed (Wapenaar et al., 2011). Wu et al (2014) proposed a direct nonlinear inversion using inverse thin-slab propagator for a tomographic problem. However the method assumed the T-matrix is known. The traditional approach to obtain full T-matrix is by an iterative linearized inversion, in which scattering potential and T-matrix are updated alternatively. In this work we propose to use a non-perturbative method of Green's function and T-matrix reconstruction based on causal backpropagation integral which uses the background Green's function to recover the scattered wavefield. We notice that the interference from the anti-causal scattered waves is the key problem and needs to be further studied.

Green’s function and T-matrix in scattering theory

Through the Lippmann-Schwinger equation for Green's operator \( G \) (see equation (3), T-matrix can be defined as (Taylor, 1972, Chapter 8) (in matrix or operator form) \( T = V + \text{VG} \) (1)

where \( V \) is the scattering potential. For the acoustic wave equation, it can defined as

\[
V(x,x') = k^2 \int d^3x' \left( \frac{c_s^2}{c_r^2(x')} - 1 \right) \delta(x - x'), \quad x,x' \in V
\]  

(2)

By the definition, we see that T-matrix is the intrinsic scattering property of the medium depending only on \( V \). T-matrix is a convenient tool for scattering theory and experiment. With the knowledge of T-matrix, it is easy to calculate the full Green's function:

\[
G = G_s + G_s \text{VG} = G_s + G_s \text{TG}
\]  

(3)

T-matrix has also a direct connection with the scattering experiments:

\[
d(x,x,) = \delta u(x,x,) = (x | G_s \text{VG} | x) = (x | G_s \text{TG} | x) \]  

(4)

where \( d = \delta u \) is the data vector (scattered pressure field), and \((x, x,) \) is the source-receiver pair in the measurement surface. In this paper we will use the Dirac’s “bra” and “ket” notation. Historically, both equations (1) and (4) were considered as the definition of T-matrix, one from scattering theory, the other from scattering experiment. From scattering theory, in order to reconstruct the full Green’s operator \( G \), you need to define \( T \) everywhere; on the other hand, for a single frequency scattering experiment, you can only define and recover the “on-shell” T-matrix (ibid) define by (4) (in the wavenumber domain), which is in contrast with the definition of (1) where you need both “on-shell” and “off-shell” T-matrix components. We will discuss this problem further in the next section. In some literature (e.g. Weglein et al, 2006) the full T-matrix is called “generalized T-matrix”. In this paper we will call it the full T-matrix or intrinsic T-matrix, while the T-matrices recovered from different experiments as approximate T-matrices.

Backpropagation integral and scattered field reconstruction (Green’s function retrieval)

Assuming there is no active sources inside the surface \( S_d \) which surrounds a volume \( V_d \), so there are only equivalent scattering sources due to heterogeneities (here velocity perturbations). Then the wave equation can be written as

\[
\left( \nabla^2 - \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \right) \delta u(x,t;x,) = \left( \frac{1}{c_r^2} - \frac{1}{c_s^2} \right) \frac{\partial^2}{\partial t^2} u(x,t;x)
\]  

(5)

where \( Q(x,t;x) \) is the scattering potential. For the acoustic wave equation, it can defined as

\[
V(x,x') = k^2 \int d^3x' \left( \frac{c_s^2}{c_r^2(x')} - 1 \right) \delta(x - x'), \quad x,x' \in V
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where $\delta u$ is the scattered field, $u$ is the total field and $Q$ is the equivalent scattering source. Assume the scattered field $d(x,t,x_s) = \delta u(x,t,x_s)$ due to the equivalent sources $Q$'s have been measured on a closed surface $S_m$, such as a sphere (see Figure 1). In order to see the physical meaning of the derivation, we consider the full-aperture acquisition in this paper, and leave the influence of limited aperture for future study. In operator form, the data operator is defined through

$$d(x,t,x_s) = \delta u(x,t,x_s) = (x, |\delta u|, x, x_s \in S_m) \quad (6)$$

The scattered field at any point inside the volume can be reconstructed by a backpropagation integral (BPI) (Bojarski, 1983; see also, Shuster, 2009;)

$$\int_{S_M} dS(x) \left[ \tilde{g}_0 \ast \frac{\partial d(x,s)}{\partial n} - d(x,x_s) \ast \frac{\partial \delta u}{\partial n} \right]$$

$$= \delta u(x,t,x_s) - \delta \tilde{u}(x,t,x_s), \quad x \in V_M \quad (7)$$

where $\ast$ denotes the time-convolution, $g_0(x,t,x_s)$ and $\tilde{g}_0(x,t,x_s)$ are the causal and anti-causal (time-advanced) Green's functions respectively, $\delta u$ is the scattered wave field, and $\delta \tilde{u}$, the anti-causal scattered field produced by the equivalent source inside $S_m$, .

$$\delta u(x,t,x_s) = \int_{V_M} g_0((x-x',t) \ast Q(x',t,x) x', x \in V_M \quad (8)$$

$$\delta \tilde{u}(x,t,x_s) = \int_{V_M} \tilde{g}_0((x-x',t) \ast Q(x',t,x) x', x \in S_M \quad (8)$$

Therefore, the new wave-field pair (causal field and its anti-causal counterpart) $\delta u(x,t,x_s) - \delta \tilde{u}(x,t,x_s)$ reconstructed by the BPI can be considered as the scattered field pair received by a receiver at $x$, excited by an active source at $x_s$ on the surface.

Figure 1. The designed acquisition of full aperture. The red star is the source and blue triangular are receivers. The red dots denote the receiver which is used for comparisons between recorded scattered wave-field and the back propagated wave-field.
Figure 3: Recorded wave-field by actual receivers: (a) total field, (b) background field, (c) scattered field.

Figure 4: Reconstructed scattered wavefield by BPI (a) and the residual after subtracting the recorded scattered field. The residuals are due to the anti-causal scattered field.

Figure 5: One example of iterative inversion for a Gaussian ball using imperfect Green's matrix (10% random error) (The data are produced using a crosshole transmission geometry).

From equation (8) we see that scattered field pairs are produced by the same equivalent sources related to the perturbation function. Therefore, it is possible to have joint inversion for all the sources to separate the causal from the anti-causal. After the separation, we can reconstruct the Green's function

\[ g(x, x_r) = g_0 + \delta g = g_0 + \delta u \] (9)

With the Green's function reconstructed, we can reconstruct T-matrix based on equation (4) or invert for the scattering potential (and related velocity perturbations) directly. The advantage of using T-matrix is to apply the efficient nonlinear inversion using the inverse thin-slab propagator (ITSP) (Wu et al., 2014). Note that even the use of iterative inversion from the reconstructed Green's function for scattering potential, the method is still non-perturbative and the convergence property should be better than the perturbation based method. Figure 5 shows one example of iterative inversion for a Gaussian ball using imperfect Green's matrix (10% random error) (The data are produced using a crosshole transmission geometry).

Another approach is the equivalent source inversion. From equation (8) we see that \( \delta u(x; t; x_r) - \delta u(x; t; x_s) \) has a linear relationship with the equivalent sources \( Q_e \), so joint or iterative inversion can be set up for \( Q_e, G \) and \( V \), which is not based on the Born approximation.

The other strategy is to apply a second BPI to reconstruct the domain Green's function (inside the volume) as shown...
schematically in Figure 6. The sources (red stars at $x_s$) and receivers (blue triangles at $x_r$) are all located on a surrounding surface of the targeted volume.

![Figure 6. Schematic diagram of the double focusing process for Green's function retrieval and T-matrix reconstruction. The sources (red stars at $x_s$) and receivers (blue triangles at $x_r$) are all located on a surrounding surface of the targeted volume.](image_url)

Denote the reconstructed wavefield by BPI as holographic field (Porter, 1970, 1982; Bojarski, 1983; Langenberg, 2002)

\[
\Theta(x) = \oint_{S_M} dS(x) \left[ \tilde{g}_s(x_s;x) \ast \frac{\partial \tilde{u}(x)}{\partial n} - \tilde{u}(x_s;x) \ast \frac{\partial \tilde{g}_s(x_s;x)}{\partial n} \right]
\]

\[
\Theta(x) = \oint_{S_M} dS(x) \left[ \tilde{g}_d(x_d;x_d) \ast \frac{\partial \tilde{u}(x)}{\partial n} - \tilde{u}(x_d;x_d) \ast \frac{\partial \tilde{g}_d(x_d;x_d)}{\partial n} \right]
= 0
\]

(10)

We see that the holographic reconstruction results in a causal-anticausal pair of solutions. Assume we can separate the causal solution from the anti-causal one. We denote the reconstructed causal solution by holographic reconstruction as $\Theta' (x,t;x_s)$, then we have

\[
\Theta' (x,t;x_s) = \tilde{u}(x_s;x,t), \quad x \in V_M
\]

(11)

Due to the reciprocity,

\[
\tilde{u}(x_s,t;x) = \tilde{u}(x,s,t)
\]

(12)

we can treat the reconstructed scattered field as a new data set as excited by active sources at $x_s$ and received by receivers on the surrounding surface. Then we can form a new data set on the closed surface $S_M$, and repeat the BPI, resulting in

\[
\oint_{S_M} dS(x) \left[ \tilde{g}_s(x_s;x) \ast \frac{\partial \tilde{u}(x_s;x)}{\partial n} - \tilde{u}(x_s;x) \ast \frac{\partial \tilde{g}(x',x_s)}{\partial n} \right]
= \delta u(x',t;x) - \delta \tilde{u}(x',t;x)
\]

(13)

Again the key is the separation or removal of anti-causal scattered waves from the causal ones. However, in this case the interaction is more localized. We will discuss more in the presentation.

**Conclusion**

We propose to apply the Bojarski equation for the Green's function and T-matrix reconstruction using surface data, in which only background Green's function is known. No weak-scattering assumption is needed for scattered field reconstruction. Direct nonlinear inversion can be applied to the reconstructed new data sets. The key problem is the removal of anti-causal scattered waves. We discussed different approaches for this issue.

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REFERENCES


