T-matrix representation of the De Wolf series for modeling and inversion in strongly scattering media
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SUMMARY

We report from the development of a T-matrix representation of the De Wolf series for seismic waveform modeling and inversion based on the scalar wave equation in the frequency domain. We show that the renormalized DWS has a much large convergence radius and speed than the naive Born series. We have not renormalized the inverse problem, but we have used the DWS to develop a Gauss-Newton consistent method for FWI that does not require a full forward simulation at each iteration. Since the DWS accounts for multiple backscattering as well as the phase-accumulation in the forward direction, the forward model can be accurately updated after each iteration via simple matrix multiplication independent of the source-receiver geometry. The excellent convergence properties of the DWS are illustrated with numerical examples dealing with seismic modeling and FWI in strongly scattering media.

INTRODUCTION

Seismic modeling and inversion are essentially scattering and inverse scattering problems similar to the ones that have been studied in other parts of physics and engineering for many years (Newton, 1992; Pike and Sabatier, 2002). Therefore, it may be a good idea to modify the highly developed multiple scattering and renormalization methods that have been developed to solve scattering and inverse scattering problems in physics for use in seismics (Jakobsen and Ursin, 2015). In this paper, we report from the development of a renormalized multiple scattering theory that can be used for both modeling and inversion. By renormalized, we mean that the different terms in the well-known Born series have been reorganized and partially summed, so that their order better reflect the physics of seismic wave propagation, especially related with reflection seismology (Wu et al., 2007, 2014a,b).

The Born series may not converge in the presence of large contrast volumes (Innanen, 2009). This is a problem since the (distorted) Born approximation is used to calculate the sensitivity matrix in standard full waveform inversion (Virieux and Operto, 2009). The De Wolf series has much better convergence properties, since it accounts for all multiple scattering in the forward direction and as many back-scattering terms as required (Wu et al., 2007, 2014a,b). However, only the first term of the De Wolf series, the so-called De Wolf approximation has so far been implemented in the context of reflection seismology (Wu et al., 2007). On the basis of the works of Wu et al. (2007) and Jakobsen (2012), we have developed an efficient T-matrix representation of the De Wolf series that can be useful for both modeling and inversion. The T-matrix representation of the De Wolf series represents a renormalization of the direct forward scattering problem, that can also be used to eliminate the need to perform a full forward simulation at each iteration in scattering-based FWI (Jakobsen and Ursin, 2015).

ACTUAL AND BACKGROUND GREEN FUNCTIONS

The Green function for the scalar wave equation in the frequency domain (the Helmholtz equation) satisfies (Morse and Feshbach, 1953)

$$\left( \nabla^2 + \frac{\omega^2}{c^2(x)} \right) G(x,x') = -\delta(x-x'),$$

where $c(x)$ is the wave speed at position $x$ and $\omega$ is the angular frequency. Defining $c_0(x)$ as the wave speed in an arbitrary heterogeneous background medium, we get

$$\left( \nabla^2 + \frac{\omega^2}{c_0^2(x)} \right) G(x,x') = -\delta(x-x') + \omega^2 P(x)G(x,x'),$$

where

$$P(x) = c^{-2}(x) - c_0^{-2}(x)$$

is the perturbation in the squared slownesses. The second term on the right-hand side of equation (2) represents the so-called contrast-sources. By using the same source representation theorem for both real and (virtual) contrast-sources, we obtain (Morse and Feshbach, 1953; Jakobsen and Ursin, 2015)

$$G(x,x') = G^{(0)}(x,x') + \omega^2 \int_{\Omega} dx''G^{(0)}(x,x'')P(x'')G(x'',x'),$$

where $G^{(0)}(x,x')$ is the background medium Green function, that satisfies

$$\left( \nabla^2 + \frac{\omega^2}{c_0^2(x)} \right) G(x,x') = -\delta(x-x').$$

If the background medium is homogeneous then one can use simple analytical expressions for $G^{(0)}(x,x')$. If the background medium is not homogeneous then one can of course use ray theory or the finite difference method to estimate or compute the background medium Green function. In this study, however, we use the above equation in conjunction with the T-matrix approach (discussed below) to relate homogeneous and heterogeneous background media Green functions (see also Jakobsen and Ursin, 2015).

For compatibility with Dirac’s bra-ket notation for linear integral operators (Taylor, 1972), the Dyson equation (4) can be rewritten exactly in the form of a product of continuous matrices (Jakobsen and Ursin, 2015):

$$\bar{G}(x,x') = \bar{G}^{(0)}(x,x') + \int_{\Omega} dx_1 dx_2 \bar{G}^{(0)}(x_1,x_2)V(x_1,x_2)\bar{G}(x_2,x'),$$

where the bar above the (modified) Green functions indicate that they have been multiplied with $\omega^2$, and

$$V(x,x') = P(x)\delta(x-x').$$

is a local scattering potential. By absorbing the $\omega^2$ factor of the contrast source into the modified Green functions, we obtain a frequency-independent scattering potential suitable for inversion (see Kouri and Vijay, 2003).
THE TRANSITION OPERATOR FORMALISM

The symmetric form (6) of the Dyson equation (4) can be regarded as the real-space coordinate representation of an operator equation for the actual and background media Green function operators:

\[ \tilde{G} = \tilde{G}^{(0)} + \tilde{G}^{(0)} V \tilde{G} , \quad (8) \]

or equivalently (Newton, 2002)

\[ G = G^{(0)} + G V G^{(0)} . \quad (9) \]

An advantage of the operator formalism is that the equations and derivations are independent of the representation, so we can focus on the physics and main principles. Following the quantum mechanical potential scattering approach (Taylor, 1972; Weglein et al., 1981; Jakobsen, 2012), we now introduce a transition operator \( T \) that relates \( \tilde{G} \) and \( G \) by

\[ V \tilde{G} = T \tilde{G}^{(0)} \quad (10) \]

From equations (8) and (10) and the fact that the background medium is arbitrary, it follows that (Taylor, 1972; Newton, 2002; Kouri et al., 2003; Jakobsen, 2012)

\[ T = V + V G^{(0)} T . \quad (11) \]

Thus, the T-operator satisfies an integral equation of the Lippmann-Schwinger type, independent of the source-receiver configuration. Equation (11) has the following formal solution

\[ T = V (I - G^{(0)} V)^{-1} , \quad (12) \]

which corresponds to the inversion of a huge matrix in the real-space coordinate representation (Jakobsen and Ursin, 2015). In principle, \( T \) can also be evaluated using the following Neumann series:

\[ T = V \sum_{m=0}^{\infty} (G^{(0)} V)^m . \quad (13) \]

However, this naive Born expansion is only guaranteed to converge if the norm of the operator \( \tilde{G}^{(0)} V \) is smaller than unity. In what follows, therefore, we shall introduce a renormalization scheme based on the so-called scattering-path operator.

THE SCATTERING-PATH OPERATOR

An arbitrary scattering domain with complete scattering potential \( V \) can always be decomposed into \( M \) (an arbitrary integer) sub-domains with scattering potentials \( V_m \), so that

\[ V = \sum_{m=1}^{M} V_m . \quad (14) \]

If we assume that the corresponding T-operator can be written

\[ T = \sum_{m=1}^{M} T_m . \quad (15) \]

then it follows from the Lippmann-Schwinger equation (11) that relates the \( V \)- and \( T \)-operators that

\[ T_m = V_m + V_m \tilde{G}^{(0)} \sum_{n=1}^{M} T_n . \quad (16) \]

The above equation (16) can be rewritten exactly as (Gonis and Butler, 1990; Jakobsen, 2012)

\[ T_m = t_m + \sum_{n=1}^{M} t_m G^{(0)} (1 - \delta_{mn}) T_n . \quad (17) \]

where

\[ t_m = V_m (I - \tilde{G}^{(0)} V_m)^{-1} . \quad (18) \]

Equations (15) and (17-18) represents a separation of intra- and inter-domain interaction or multiple scattering terms. The complete T-operator works on all space, but \( t_m \) is a restricted (self-interaction) operator that works only on the \( m \)-th domain. In other words, computation of the small t-operators is characterized by a relatively small computational cost compared to the computation of the full T-operator. As discussed below, the physical interpretation in terms of inter- and intra-domain interaction terms can be very useful when trying to derive a renormalized scattering series.

The domain decomposition of the T-operator represented by equations (15) and (17-18) is equivalent to the following expression:

\[ T = \sum_{m=1}^{M} \sum_{n=1}^{M} \tau_{mn} . \quad (19) \]

where

\[ \tau_{mn} = t_m \delta_{mn} + \sum_{k=1}^{M} t_m G^{(0)} (1 - \delta_{mk}) \tau_{kn} . \quad (20) \]

Here, \( \tau_{mn} \) is the so-called scattering path operator (SPO) that accounts for all scattering events between domains \( n \) and \( m \). The SPO is commonly used in connection with the Korrings-Kohn-Rostoker Green function method of solid state physics (Gonis and Butler, 1990), but we have not seen any use of this general concept in connection with seismic scattering.

The decomposition of the T-operator into a sum of all scattering-path operators is valid for any decomposition of the medium. However, the concept of a scattering-path operator becomes particularly transparent and useful if we decompose the medium into a set of thin horizontal layers (thin-slabs) with laterally variable velocities. T-matrices corresponding to up- and down-going waves can then in principle constructed by restricting the sums in equation to \( m > n \) and \( n < m \), respectively. However, it is more efficient to use to exact result for both domains in the recursive manner detailed below.

RECURSIVE SCHEME FOR ONE-WAY PROPAGATORS

From equation (19), we find that the exact result in the case of two interacting domains can be written as

\[ \tau_{11} = t_1 + t_1 \tilde{G}^{(0)} \tau_{21} , \quad (21) \]

\[ \tau_{12} = t_1 \tilde{G}^{(0)} \tau_{12} , \quad (22) \]

\[ \tau_{21} = t_2 \tilde{G}^{(0)} \tau_{12} , \quad (23) \]

\[ \tau_{22} = t_2 + t_2 \tilde{G}^{(0)} \tau_{22} . \quad (24) \]

The above result is exact but explicit; that is, not directly suitable for recursive application to multiple layers. In the case of
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an up- and down-going waves, however, we have \( \tau_{21} = 0 \) and \( \tau_{12} = 0 \), respectively. Therefore, the exact result for two domains becomes explicit and very simple in the case of one-way propagators:
\[
T^{(u,2)} = t_1 + t_2 + t_1 G^{(0)} t_2 \tag{25}
\]
\[
T^{(d,2)} = t_1 + t_2 + t_2 G^{(0)} t_1. \tag{26}
\]
Since the first domain is arbitrary, we can add the different thin-slabs one by one in a sequential manner, and then update the \( T^{(u)} \) - and \( T^{(d)} \)-operators after then addition of each new thin slab using the following novel recursive scheme:
\[
T^{(u,n)} = T^{(d,n-1)} + t_n + T^{(d,n)} G^{(0)} t_n, \tag{27}
\]
\[
T^{(d,n)} = T^{(u,n-1)} + t_n + t_n G^{(0)} T^{(u,n-1)}. \tag{28}
\]
In the case of \( N \) layers or thin-slabs, the overall T-matrices for up- and down-going waves are given by
\[
T^{(u)} = T^{(u,M)}, \tag{29}
\]
\[
T^{(d)} = T^{(d,M)}. \tag{30}
\]
respectively. The corresponding scattering potentials for up- and down-going waves are given by
\[
V^{(u)} = T^{(u)} - T^{(u)} G^{(0)} V^{(u)} \tag{31}
\]
\[
V^{(d)} = T^{(d)} - T^{(d)} G^{(0)} V^{(d)} \tag{32}
\]
Due to the triangular nature of the \( T^{(u)} \) - and \( T^{(d)} \)-operators, the corresponding \( V^{(u)} \) - and \( V^{(d)} \)-operators can be found via back-substitution, without having to invert a full domain integral operator. Alternatively, one can use the inverse thin-slab method of Wu et al. (2014) to determine the (non-local) scattering potentials \( V^{(u)} \) and \( V^{(d)} \).

THE DE WOLF SERIES (DWS)

In order to derive a T-operator version of the De Wolf series (Wu et al., 2007), we now assume that the total scattering potential \( V \) operator can be decomposed as
\[
V = V^{(f)} + \Delta V^{(b)}, \tag{33}
\]
where \( V_f \) and \( \Delta V_b \) are the parts of \( V \) that are responsible for multiple scattering in the forward and backward directions, respectively. By substituting the above expression into the Dyson equation (8) and treating \( V_f \) as a new background medium, we get
\[
\bar{G} = \bar{G}^{(f)} + \bar{G}^{(f)} \Delta V^{(b)} \bar{G}. \tag{34}
\]
where
\[
\bar{G}^{(f)} = G^{(0)} + G^{(0)} V^{(f)} G^{(0)}. \tag{35}
\]
The above equation for \( G_f \) can be rewritten exactly as
\[
\bar{G}_f = \bar{G}_0 + \bar{G}_0 T^{(f)} \bar{G}_0, \tag{36}
\]
where
\[
T^{(f)} = V^{(f)} + V^{(f)} G^{(0)} T^{(f)} \tag{37}
\]
is a T-operator that accounts for all effects of multiple scattering in the forward direction only.

Since the backscattering part is normally smaller than the forward scattering part, the renormalized Dyson equation may be solved via a Neumann expansion:
\[
\bar{G} = \sum_{p=0}^{+\infty} \left( \bar{G}_f \Delta V^{(b)} \right)^p \bar{G}^{(f)}. \tag{38}
\]

It remains to discuss how one can implement the above system of equations for a reflection seismology geometry, where forward and backward have a different meaning for upward and downward propagating waves.

T-MATRIX REPRESENTATION OF DWS

In the real-space coordinate representation, all Green operators may be represented by matrices after spatial discretization. It is possible to use the bra-ket notation to do this in a very formal manner, but here we keep the discussion and notation as simple as possible. We denote by \( G_{RS} \), \( G_{RV} \) that \( G_{VS} \) the Receiver-Source, Receiver-Volume and Volume-Source Green matrices, where the the propagation direction goes from right to left.

For seismic modeling based on the T-matrix representation of the DWS, we use a symmetric expression for \( G_{RS} \)
\[
G_{RS} = G^{(0)}_{RS} + \frac{1}{2} G^{(f)}_{RV} V G_{VS} + \frac{1}{2} G^{(f)}_{RV} V G^{(0)}_{VS}. \tag{39}
\]
that follows from the equivalence of the operator equations (8) and (9). The symmetric form (40) is particularly suitable for surface seismic reflection experiments. Here
\[
G_{VS} = \sum_{p=0}^{+\infty} \left[ \left( V - V^{(f)} \right) G^{(f)} \right]^p, \tag{41}
\]
where \( f_p \) is equal to \( u \) or \( d \) when \( p \) is an even or odd or number, respectively. Thus, we use the De Wolf series for the computation of the actual medium Green matrices \( G_{VS} \) and \( G_{RV} \) for which the concept of forward and backward scattering is well-defined. In the case of \( G_{VS} \) and \( G_{RV} \), the forward directions correspond with downward and upward propagating waves, respectively.

In the (Newton-Kantorovich-consistent) distorted Born iterative (DBI) inversion method (Chew and Wang, 1991), the data residuals at the \( i \)th iteration is linearly related to the change in scattering potential between two iterations:
\[
G_{RS} - G^{(i)}_{RS} = G^{(i)}_{RV} \left( V^{(i+1)} - V^{(i)} \right) G^{(i)}_{VS}. \tag{42}
\]
In the T-matrix variant of the DBI method presented by Jakobsen and Ursin (2015), the need to invert a huge matrix to update the background medium Green functions after each iteration of FWI was eliminated via the use of a Born series for the change in the corresponding T-matrix. In this study, we have improved the efficiency of the DBI inversion method by replacing the Born series for the T-matrix variations with the corresponding DWS, which has much better convergence properties.
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Figure 1: Example of a strongly scattering medium. Displayed is the SEG salt model, in a resampled version.

Figure 2: Test of the convergence properties of the Born series for the strongly scattering medium in Figure 1.

Figure 3: Test of the convergence properties of the Born series for the strongly scattering medium in Figure 1.

NUMERICAL EXAMPLES

In order to compare the relative performance of the distorted Born and De Wolf series for seismic modeling and inversion, we now consider the SEG salt model as an example of a strongly scattering medium. To keep the computational cost at a reasonable level, we resample the SEG salt model using a grid size equal to 24 m in each direction. The resampled version of the SEG salt model (with 115 times 24 grid blocks) we have used for modeling is shown in Figure 1, together with a depth-dependent background velocity model. In Figures 2 and 3, we present synthetic waveform data for the central frequency 7.5 Hz (corresponding to a single source in the middle of the model), constructed using the distorted Born and De Wolf series, respectively. Clearly, one can see that the De Wolf series works much better than the Born series for the strongly scattering medium in Figure 1.

For full waveform inversion in the frequency domain using the distorted Born iterative method, we employ a Ricker wavelet with a central frequency equal to 7.5 Hz in conjunction with a resampled SEG salt model. The model we use in the inversion examples is similar to the one in Figure 1, but we now use 69 times 14 grid blocks to reduce the computational time. In

Figure 4 one can see the true model, the start model and two inverted models obtained using the exact T-matrix and a 9-order De Wolf approximation, respectively.

Figure 4: Use of the T-matrix representation of the De Wolf series in conjunction with the distorted Born iterative inversion method. The upper left and right figures show the true and start models, respectively. The lower left and right figures show the inverted models obtained using the exact T-matrix and a 9-order De Wolf approximation, respectively.

CONCLUDING REMARKS

We have developed a T-matrix representation of the De Wolf series method for modeling of surface seismic reflection data. The new scheme is based on the use of the scattering-path operator, which was used to construct a fast recursive scheme for one-way propagators. Numerical experiments confirm that the convergence radius and speed of the De Wolf series is superior to the Born series. The work presented here may find use in different modeling problems, and also be important for the development of more direct inversion methods based on a renormalized scattering series.
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EDITED REFERENCES
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