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Elastic One-Return Boundary Element Method and Hybrid Elastic Thin-Slab Propagator

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SUMMARY

We developed the theory and algorithm of an elastic one-way boundary element method and a corresponding hybrid elastic thin-slab propagator for strong-contrast media with sharp boundaries. It takes the advantage of accurate boundary condition of BEM (boundary element method) and completely overcomes the weak contrast limitation of the perturbation-theory based one-way operator approach. The one-way BEM is a smooth boundary approximation, which avoids huge matrix operations in exact full BEM; in addition, the one-way BEM can model the primary-only transmitted and reflected waves and therefore is a valuable tool in elastic imaging and inversion. Through numerical tests for some simple models, we proved the validity and efficiency of the proposed method.
Introduction

The elastic one-way wave method has some special features and advantages in the application to seismic modeling and imaging/inversion. One-way wave method in general is much more efficient, often orders of magnitude faster than the full wave method; it also requires less internal memory in the calculation. Secondly, one-way method can model primary waves, including primary transmitted and reflected waves, which are difficult to isolate in full wave modeling. In addition, the elastic one-way method has a number of useful attributes including the ability to switch on and off the mode conversions on sharp boundaries, so that different converted wave paths can be separated. These special features are useful for elastic wave imaging, inversion and multiple removals.

However, existing elastic one-way wave methods have shortcomings which are critical for applications involving strongly heterogeneous media with sharp boundaries. Most elastic one-way methods are based on the perturbation theory (e.g., Landers and Claerbout, 1972; Hudson, 1980; Wapenaar and Berkhout, 1989; Wu, 1994, 1996; Thomson, 1999, 2005; Xie and Wu, 2001; for a review, see Wu et al., 2007), which are not applicable to strongly heterogeneous media. Most perturbation-based method can handle elastic perturbations only up to 30%. The algorithms may become inaccurate and unstable beyond this limit. It is pressing to extend the one-way elastic method to the case of strong-contrast media.

On the other hand, the BIE (boundary integral equation) and BEM (boundary element method) can handle sharp boundaries of blocked smooth media (for a review, see Bouchon and Sanchez-Sesma, 2007). The problem there is the lack of an efficient method in calculating the Green's functions in weakly heterogeneous media, thereby preventing its general use in strongly heterogeneous media. In addition, computation cost in solving the huge matrix equations involved can be insurmountable in many cases. Therefore it is attractive to develop a hybrid one-way, one-return operator which combines the boundary element treatment for the sharp boundaries and the perturbation approach for weak heterogeneities. He and Wu (2007) proposed a BEM method in which the top-salt boundary and the bottom salt boundary are decoupled, so that the primary reflections and multiples due to salt boundary can be separated. Yan et al., (2013) tested a one-way elastic hybrid propagator in high-contrast media using the physical optics approximation for boundary scattering. In this study, we develop the theory and method of a hybrid elastic one-way operator which combines the efficiency of elastic generalized screen propagator in weakly heterogeneous media and the accuracy of BEM at the sharp boundaries for strong contrast media.

BIE (boundary integral equation) and BEM (boundary element method) and their one-way approximation

Assume a volume $V$ is surrounded by a surface $S$ (Figure 1). Boundary integral equation (BIE) can be formulated when the calculation point $r$ is located on the boundary surface $S$. As shown in Figure 1, we formulate a thin-slab propagator based on the BIE. Without losing generality, we treat the right segment of the boundary $S_1$, which separate the thin-slab into two regions (domains), $\Omega_1$ (exterior) and $\Omega_2$ (interior). First we calculate the scattered field generated by the boundary element $S_1$. Assume we know already the field on the plane at depth $z_0$ (entrance of the thin-slab), the incident wave at any point in the thin-slab can be calculated by the surface representation integral (the elastic Kirchhoff integral) along the right-hand-side segment (or the exterior disk in the 3D case). We also assume that the edge effect at the right end can be removed or be neglected. The incident wave has included the scattered waves by all the previous boundary elements above the current one; therefore it might be considered as the Green's function calculation in heterogeneous media. In the spirit of one-way, one-return approximation, we write the integral equation system for the $u(r) = u_0(r) + u_1(r)$ at the boundary between $\Omega_1$ and $\Omega_2$, as

$$
\int_{S_1} [f_1(r, \vec{r}_1) \cdot G_1(r, r') - u_0(r) \cdot \Gamma_1(r, r'; \vec{n}_1)] dS(r') = \sigma(r) u_1(r), \quad r', r \in \partial \Omega_1
$$

$$
\int_{S_1} [f_1(r, \vec{r}_2) \cdot G_2(r, r') - u_0(r) \cdot \Gamma_2(r, r'; \vec{n}_2)] dS(r') = \sigma(r) u_1(r), \quad r', r \in \partial \Omega_2
$$

(1)
where \( a(r) \) is a constant depending the position of the point and the local geometry. When the point is a corner point, \( a(r) \) represents the portion of the angle span of local boundary segments. For flat boundaries, it takes the values 0.5; \( t \) and \( u \) are the traction and displacement reflected fields, i.e. the scattered fields, \( G_1 \), \( T_1 \) are the Green’s displacement and tractions in \( \Omega_1 \), and \( G_2 \), \( T_2 \) are those in \( \Omega_2 \). Here \( \hat{n}_1 \) and \( \hat{n}_2 \) are the outward normals in domains 1 and 2, respectively. The total fields \( t(r') \) and \( u(r') \) are equal to the transmitted fields in \( \Omega_2 \) in this case. In the representation integral, the traction force is defined with respect to the outward normal. Therefore, the direction of the total traction in \( \Omega_2 \) is opposite to that in \( \Omega_1 \) but with the same magnitude due to the traction continuity across the boundary, i.e. \( t(r'; \hat{n}_2) = -t(r'; \hat{n}_1) \). The above equation is to be solved with boundary condition and the known incident field \( u_0 \) and \( t_0 \).

Neglecting the horizontal interaction between \( S_1 \) and \( S_2 \), the integral in equation (1) contains only a self-influencing element. This neglect will exclude the horizontally propagated waves. If we adopt a linear element, the equations can be simplified to

\[
t_1(r, \hat{n}_1) \cdot g_1 - u_1(r) \cdot \gamma_1 = a(r) u_1(r) \\
-t_1(r, \hat{n}_1) \cdot g_2 - u_1(r) \cdot \gamma_2 = a(r) u_1(r)
\]

(2)

where

\[
g_{1,2}(r) = \int_{S} [L(r' - r)G_{1,2}(r'; r')dS(r')] \\
\gamma_{1,2}(r) = \int_{S} [L(r' - r)T_{1,2}(r'; r')dS(r')]
\]

(3)

where \( L(r' - r) \) is the linear shape function of the element. Equation (2) can be solved by simple matrix operation.

**Figure 1** Schematic illustration of the one-way boundary element derivation. A thin-slab is marked by the yellow color.

**Hybrid boundary-volume scattering thin-slab propagator for elastic one-return method**

In a piecewise heterogeneous medium with sharp boundaries, we can divide the medium into blocks (domains). The strong boundary scattering will be handled by the one-return BEM method described in the above section. The weak heterogeneities within each blocks are treated by the perturbation method as in the generalized screen propagator (e.g., Wu et al., 2007). We develop a dual-domain implementation: propagation in the wavenumber domain and interaction with heterogeneities in the space domain. However, both the free propagation (background propagation) and heterogeneity interaction in the hybrid propagator are different from the perturbation-based propagator as discussed in the following.
At the exit of each thin-slab, the total field is composed of three parts: the incident (free-propagated) wave \( u_i^{(0)} \), the boundary scattered field \( u_{B_i}^{(0)} \) and the scattered field by volume heterogeneities \( u_{V_i}^{(0)} \),

\[
    u^{(0)} = u_i^{(0)} + u_{B_i}^{(0)} + u_{V_i}^{(0)} \tag{4}
\]

Comparing to the perturbation theory based one-way propagator, the scattered waves of boundary scattering is an extra term for the contributions from individual boundary elements in the domain \( i \) of the current thin-slab. This is still a single-scattering approximation within the thin-slab, and is consistent with the De Wolf approximation. However, this boundary scattering term is critical for the accuracy and stability in strong-contrast media.

**Numerical examples of one-way BEM**

We compare the modeling accuracy of one-way BEM with the full BEM for a simple model: a high-velocity round ball. The background is homogeneous with \( V_p = 2.5 \text{ km/s}, V_s = 1.5 \text{ km/s} \) and \( \rho = 2.0 \text{ g/cm}^2 \). The ball has a radius of \( r = 1.5 \text{ km}, V_p = 4 \text{ km/s}, V_s = 2.5 \text{ km/s} \) and \( \rho = 3.0 \text{ g/cm}^2 \). Space-domain exact Green's function in homogeneous media is used for both full-wave and one-way BEM’s. In the full-wave BEM, the boundary of the ball is discretized into 1080 equally sized elements, so the BE matrix is about \( 4000 \times 4000 \). An explosion source is located at \( x = 3.5 \text{ km}, z = 0 \) with a Ricker source wavelet centered at \( f = 10 \text{ Hz} \). The maximum frequency calculated is 25 Hz with a 0.25Hz interval. So the total number of frequency used is 100. For the highest frequency 25 Hz, the shortest wavelength (S-wave) is 60 m, and the element size is about 10 m. Therefore, the element size is small enough for accurate BEM calculation. Figure 2 shows the snapshots at \( t = 1.3 \text{ sec} \) calculated by both methods. The calculation is carried out in a workstation. Since the full BEM involves huge matrix operations, it takes about 10 hours; meanwhile the one-way BEM took less than 20 minutes. In the one-way BEM calculations, we include only the transmitted waves. If we compare the P and S wavefronts inside the ball, we see the strong similarity between the results of these two methods. This provides the basis for the use of one-way BEM in imaging and inversion.

A model of elliptical high-velocity inclusion is also used for the tests. The background velocities of P and S wave are \( \alpha = 2.5 \text{ km}, \beta = 1.5 \text{ km/s} \), while the inclusion’s P and S velocities are \( \alpha = 4.0 \text{ km}, \beta = 2.5 \text{ km/s} \). The snapshots at 1.1s are shown in Figure 3. The converted and transmitted waves can be clearly seen.

**Conclusion**

We developed the theory and algorithm of an elastic one-way boundary element method and a corresponding hybrid elastic thin-slab propagator for strong-contrast media with sharp boundaries. The fundamental basis of this operator is still the one-way approximation which neglects the reverberations (multiples along the preferred direction). However, the accuracy and applicability of the new one-way and one-return operator built upon the boundary element approach are expected to be much improved compared to the traditional perturbation approach. The new one-way method is supposed to work for strong-contrast media and large angle scattering. Through numerical tests, we have proved the validity of our proposed theory and method.

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**References**


Figure 2: Comparison of the snapshots (at t = 1.3 sec) from one-way BEM (bottom) and Full wave BEM (top) for a high-velocity round ball. On the left are for the horizontal component (ux) and on the right, vertical component.

Figure 3: The snapshot for the high-V inclusion model at 1.1s calculated by the one-way BEM. The converted and transmitted waves can be clearly seen.