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Nonlinear Sensitivity Operator and Its De Wolf Approximation in T-matrix Formalism

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SUMMARY

We derived the Nonlinear sensitivity operator in nonlinear tomographic waveform inversion based on the theory of nonlinear partial derivative operator. We apply the renormalization procedure (De Wolf approximation) to the forward and inverse T-matrix series. Numerical tests proved that the renormalized inverse scattering series has much better convergence property than the inverse Born series. This convergence improvement may be applied to the iterative procedure of waveform inversion.
Introduction

Current gradient method in full waveform inversion (FWI) is based on a linearization of the full nonlinear functional partial derivative (NLPD) operator, and can be considered as a quasi-linear inversion. NLPD can be expanded into a Taylor series which corresponds to a full scattering series, the Born series. The convergence problem of the iterative procedure of quasi-linear inversion, problem of local minima, and the starting model dependence, are all deeply rooted in the well-known convergence problem of the Born series and inverse Born series (see e.g., Morse and Feshbach, 1953; Prosser, 1969; Weglein et al., 2003). For the real Earth, the wave equation is strongly nonlinear with respect to the medium parameter changes. Wu and Zheng (2012, 2014) introduced the higher order Fréchet derivatives and the theory of nonlinear partial derivative (NLPD) operator for the acoustic wave equation. Our precious work (Wu and Zheng, 2012, 2013; Wu et al., 2013) have reported the renormalization procedure using De Wolf series and approximation to improve the convergence of forward scattering series. In this work we will first introduce the De Wolf approximation to the T-matrix series and report the progress in improving the convergence of inverse Born series by renormalization procedure in the form of T-matrix series.

Summary on nonlinear partial derivative operator.

Assume an initial model \( m_0 \), we want to quantify the sensitivity of the data change \( \delta d \) (also called “data residual”) to the model perturbation \( \delta m \),

\[
\delta d = d - d_0 = F_{m_0}(\delta m) = A(m_0 + \delta m) - A(m_0)
\]

If we define a nonlinear partial derivative operator \( A^{(NLPD)}(m_0, \delta m) \) based on the nonlinear differential operator \( F(m_0, \delta m) \) through \( A^{(NLPD)}(m_0, \delta m) \delta m = F(m_0, \delta m) \), then we have

\[
A^{(NLPD)}(m_0, \delta m) = \Lambda' (m_0) \delta m + \frac{1}{2!} \Lambda'' (m_0) (\delta m)^2 + \cdots + \frac{1}{n!} \Lambda^{(n)} (m_0) (\delta m)^n + \cdots.
\]

where \( \Lambda' \), \( \Lambda'' \), and \( \Lambda^{(n)} \) are the first, second, and the \( n \)th order Fréchet derivatives. Note that \( A^{(NLPD)} \) is \( \delta m(x) \)-dependent because of the nonlinear mutual interactions) between perturbations

If we split the scattering operator into forward scattering and backscattering parts, \( S = S_f + S_b \) and substitute it into the Fréchet series, we can have all combinations of higher order forward and backward derivatives. The De Wolf approximation corresponds to neglecting multiple backscattering (reverberations), i.e. dropping all the terms containing two or more backscattering operators but keeping all the forward scattering terms untouched (De Wolf, 1985; Wu, 2003).

Forward and inverse scattering series in T-matrix approach

IBS (inverse Born series) originally is formulated directly in the data space. An alternative way, which is more convenient and computational efficient, is to formulate the inverse scattering in the image space (model space). This is the approach of contrast-source approach and T-matrix approach (e.g., Prosser, 1969; Weglein et al., 2003; Jakobsen, 2012; Jakobsen and Ursin, 2012). Here, we will apply the T-matrix approach to the NLSO (nonlinear sensitivity operator).

T-matrix (transition matrix) \( T \) is defined through the following equation

\[
\delta d(x, x') = p_s(x, x') = G_0(x, x) V(x, x') g(x', x) = G_0(x, x) T(x, x') g_0(x', x)
\]

where \( p_s \) is the scattered field, \( G_0 \) is the background Green’s operator, \( g = p \) is the total filed, \( g_0 = p_0 \) is the incident field, and \( V(x, x') \) is the scattering potential defined as (in the scalar wave case)

\[
V(x, x') = S_r(x, x') e_r(x') = k^2 \int_{V} d^3 x' e_r(x') \delta(x - x'), \quad x, x' \in V
\]

In the following we formulate the forward scattering and inversion scattering in T-matrix. First we discuss the forward problem. From the relation \( T g_0 = V g = V g_0 + V g_0 T g_0 \) we have

\[
T = [1 - V g_0]^{-1} V = PV
\]

where \( P \) is the propagator. This is in a form of integral equation similar to the Lippmann-Schwinger equation. From the above equations, we can write

\[
T = V + V g_0 T
\]
In the case of weak scattering, in which the norm $|VG_0| < 1$, an iterative procedure can be used to get a Born series of $T$-matrix from (5) for the forward scattering solution. For strong perturbations, we can apply the forward scattering renormalization, or the De Wolf approximation to eliminate the divergence problem. In the following we derive the De Wolf approximation for the $T$-matrix formulation.

**Renormalization of the $T$-matrix series and the De Wolf approximation**

We now split the scattering potential $V$ into a forward scattering part $V_f$ and a backscattering part $V_b$:

$$V = V_f + V_b$$  \hspace{1cm} (5)

Next, we write the $T$-matrix as the sum of two terms, i.e. $T = T_1 + T_2$. From (5) we have

$$T = T_1 + T_2 : \quad T_1 = V_f + V_f G_0 T : \quad T_2 = V_b + V_b G_0 T$$  \hspace{1cm} (5)

We see that $T_1$ and $T_2$ represent disturbed $T$-matrices due to forward and backward scattering, including terms related to the interaction between forward and backward scattering.

By using the renormalization procedure, one can rewrite the above formulae for the total $T$-matrix exactly as

$$T_1 = t_f + t_f G_0 T_1 : \quad T_2 = t_b + t_b G_0 T_1$$  \hspace{1cm} (5)

where the $t$-matrices stand for pure multiple-forward and pure multiple-backward scattering (reverberation) and satisfy the following integral equations

$$t_f = V_f + V_f G_0 t_f : \quad t_b = V_b + V_b G_0 t_b$$  \hspace{1cm} (5)

Various approximations can be obtained if one iterates on the above equations. To derive an approximation which accounts for all forward multiple scattering, but only single backward scattering (De Wolf approximation), we ignore multiple backward scattering (reverberations) in the second equation in (5), i.e. $t_b = V_b$. Then we have

$$T_2 = V_b + V_b G_0 T_1$$  \hspace{1cm} (5)

To first order in $V_b$, equation (5) is approximately equivalent to

$$T_1 = t_f + t_f G_0 V_b + t_f G_0 V_f G_0 t_f : \quad T_2 = V_b + V_b G_0 t_f$$  \hspace{1cm} (5)

Therefore, the total $T$-matrix under single-backscattering approximation is given by

$$T_{DeW} = T_f + T_b : \quad T_f = t_f t_f G_0 V_b + V_b G_0 t_f + t_f G_0 V_f G_0 t_f$$  \hspace{1cm} (5)

This form of De Wolf approximation for $T$-matrix is consistent with the De Wolf approximation for the wavefield.

Now we discuss the inversion through $T$-matrix. After obtaining the $T$-matrix based on data of experiments, we can invert it for the scattering potential and perturbation function $v$. The scattering data are kept in the $T$-matrix and stay in the model space (image space), and the acquisition process is peeled off. Of course, the knowledge of acquisition process is needed to estimate the $T$-matrix. By assuming the first order estimate $V_i = T$, a series solution ISS (inverse scattering series) is obtained by an iterative process:

$$V = T - VG_0 T$$  \hspace{1cm} (6)

$$V = (1 + G_0 T)^{-1} T = T \sum_{n=0}^{\infty} (-1)^n G_0 T^n = V_0 + V_1 + V_2 + ... + V_n + ...$$  \hspace{1cm} (7)

**Renormalization of ISS (inverse scattering series) under the De Wolf approximation**

Now we apply the operator split and forward-scattering renormalization to the above inverse $T$-matrix series. As we showed above, the $T$-matrix is split into a part due to forward scattering and a part due to single backscattering (neglecting multiples). In this work, we treat only the forward scattering problem such as in the case of smooth media, so only $T'$ is involved. For $T$-matrix due to forward scattering, the $T$-matrix for any point $x$ in the medium can be decomposed into one derived from the interaction with the upper half-space velocity potential (up-scattering) and one from the lower half-space velocity potential (down-scattering), plus a part from the same level.
\[
T'(x, x') = T_0(x, x') + T_{f_1}(x, x') + T_{f_2}(x, x') = T_0(x, x') + T_{f}(x, x' + T_{f}(x, x'),
\]
\[
\begin{align*}
T_{0}(x, x) &= T_0(x, x), \quad x' < z \\
T_{f}(x, x') &= T_0(x, x') + T_{f}(x, x'), \quad x' = z \\
T_{f}(x, x') &= T_0(x, x'), \quad x' > z
\end{align*}
\] (7)

Since only forward scattering is involved, we can recover the velocity potential from the corresponding \(T_{0}\), \(T_{f}\), and \(T_{f}\). Substitute the decomposition (7) into the inverse T-series (7), and apply the forward scattering renormalization, resulting in a De Wolf approximation for the inverse T-matrix series. We can prove that each term of the inverse series is a Volterra type integral (Tricomi, 1985; Schetzen, 1980). Therefore, the inverse T-series is a Volterra series which converges absolutely and uniformly (ibid).

**Numerical tests for renormalized inverse T-matrix series in comparison with the Born T-series.**

In the following we show some simple examples to demonstrate the convergence improvement of the renormalized forward and inverse T-series.

Figure 1 Left: The model of Gaussian ball \( (a = 5\lambda) \) with different perturbation strengths; Middle & Right: The full T-matrix for the model in column-vector representation: Mid: real part; Right: imaginary part.

The test model is a Gaussian ball \( (a = 5\lambda) \) with different perturbation strengths from the homogeneous background (Figure 1 on the left). The matrix T (operator matrix) is a complex-value frequency-dependent matrix of N by N. The whole model space is of 200 by 200 in grid size. The perturbation area is about 50 by 50, and N is about 1800. We produce the T-matrix by both exact matrix inverse method and the Born series. In the case of weak scattering, the two are the same. For strong scattering (above 20% perturbation), the Born series failed to converge and we rely on the exact inverse algorithm to produce the T-matrix. In order to exhibit the physical meaning of the T-matrix, we plot the column vectors from matrix T column-by-column. In Figure 1 we also plot the sparsely sampled column vectors of the full T-matrix. In the Figure, each small ball is a representation of the column vector, corresponding to a point spreading function (a scattering pattern) of the point scatterer. In the middle is the real part and on the right is the imaginary part. This exact T-matrix corresponds to a full-aperture measurement and contains all the information in acquired data. Normally, T-matrix is derived by a linear inversion from the data. In order to test the convergence of the inversion scattering series, we use the exact T-matrix here.

Figure 2(a) and 2(b) give the results of convergence tests. Here we only plot the convergence of the velocity value at the center of the ball, inverted from both the inverse Born series and the renormalized inverse scattering series. From Figure 2(a), we see that IBS converges at a slow rate for 15% perturbation, but diverges fast for 20% perturbation. In comparison, the renormalized ISS converges very fast for 15% and 20% perturbations. It still converges for 25% and 30%, but rather slowly for the latter (Figure 2(b)). We are still working on improving the convergence and to apply to waveform inversion.

**Conclusions**
We apply the renormalization procedure and the De Wolf approximation to the forward and inverse T-matrix series. Numerical tests proved that the renormalized inverse scattering series has much better convergence property than the inverse Born series. This convergence improvement may be applied to the iterative procedure of waveform inversion.

Figure 2 (a) Convergence comparison between inverse Born series (IBS) and the renormalized inverse scattering series. The red dotted lines are the correct perturbation values of velocity at the central point; the black curves are the convergence curves of the series; (b) Convergence tests of the renormalized inverse scattering series for the case of strong perturbations (25%)

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