Accuracy analysis of screen propagators for wave extrapolation using a thin-slab model
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Summary
One-way wave extrapolators, based on the split-step Fourier method or the generalized screen method, have been proposed recently as part of the solution to lessen the CPU and memory-size requirements of wave-equation-style migration and imaging in three dimensions (3D). The phase-space path-integral formulation and the associated vertical slowness symbol analysis provide a general and convenient framework for the estimation of the accuracy of such approximate propagators. By comparing the dispersion relations of the screen propagators in the high-frequency (h-f) limit with the leading-order asymptotics of the vertical slowness symbol, and by numerical simulations of transmission through a heterogeneous, thin slab, it is found that for weak perturbations with respect to a homogeneous background medium both phase-screen and wide-angle-screen propagators perform well; for strong perturbations, our modification of the wide-angle-screen propagator shows superior accuracy.

Introduction
The computational complexity of 3D one-way wave extrapolations prevents one from using wave-equation-style pre-stack depth migration, in practice. Approximations of these extrapolators, such as the split-step Fourier approach (Stoffa et al., 1990) and the generalized screen method (De Hoop and Wu, 1996) including the phase-screens, the complex-screens, and the wide-angle-screens (Wu, 1994; Wu and Xie, 1994; Wu and Huang, 1995) have been proposed recently to significantly reduce the computational complexity, at the cost of accuracy. The purpose of this paper is to analyze and improve the accuracy of the various screen methods, as a function of scattering angle. In this study, we will adopt a phase-space path-integral formulation and perform a vertical slowness symbol analysis to address the issue of angle-dependent accuracy; we will use a thin-slab model to demonstrate how well the various screen propagators behave.

Path-integral formulation
The solution of the one-way acoustic wave equation can be represented by a Hamiltonian (phase-space or ‘dual-domain’) path integral (Fishman and McCoy, 1984; De Hoop, 1996; De Hoop and Wu, 1996):

\[
\mathcal{G}(x_T, z; x_T', z') = H(|z-z'|) \int_D \mathcal{D}(x_T', \zeta_T') \exp \left[ i \omega \int_{z}^{z'} d\zeta \{ \zeta_T' \cdot (d\zeta_T') + \gamma(x_T', \zeta_T') \} \right],
\]

where \( \zeta_T \) is the horizontal wave slowness vector and is related to the horizontal wavenumber \( K_T = \omega \zeta_T \) by \( K_T = \omega \zeta_T \), \( \gamma() \) is the Heaviside function, \( \mathcal{D}(x_T', \zeta_T') \) is a measure of the ‘width’ of path \( (x_T', \zeta_T') \), and \( P \) is a set of paths \( (x_T', \zeta_T') \) in the horizontal phase space. In Eq. (1) \( \gamma \) is the left symbol of the (square-root) vertical slowness operator \( \Gamma \), which is defined by letting the operator act on a Fourier component \( \exp(i\omega \alpha_T \cdot \mathbf{x}_T) \),

\[
\Gamma(x_T, D_T) \exp(i\omega \alpha_T \cdot \mathbf{x}_T) = \gamma(x_T, \alpha_T) \exp(i\omega \alpha_T \cdot \mathbf{x}_T),
\]

where \( D_T = (-i/\omega)\nabla_T \). Let \( \rho_0, \kappa_0, c_0 \) describe the background medium and \( \rho, \kappa, c \) the actual medium, and let \( \epsilon \) denote relative perturbation, then

\[
\gamma^2 = -D_T^2 + c_0^{-2} - c_0^{-2} \mathcal{E}_p - c^{-2} \mathcal{E}_\kappa - \rho^{-1} \rho_0 [(D_T \mathcal{E}_p) \cdot D_T + (D_T^2 \mathcal{E}_p) - \rho^{-2} \rho_0^2 (D_T \mathcal{E}_p)^2]).
\]

In the case of lateral homogeneity, \( y \) does not depend on \( x_T \); then \( y = \gamma(\alpha_T) = [(1/c)^2 - \alpha^2]^{1/2} \) becomes the standard vertical slowness and the path-integral (1) reduces to the well-established phase-shift extrapolator.

If the vertical slowness left symbol (VLSLS) \( \gamma(x_T, \alpha_T) \) would be known, we would be able to march the one-way wave forward in depth precisely, although the process might be computationally very intensive. Though closed-form solutions are known for special medium profiles, in general inhomogeneous media, only asymptotic and uniform expansions of the VLSLS exist. We will use an asymptotic expansion for our accuracy analysis, the leading order of which is representative for the arrival times of the waves.

Screen approximations of the VLSLS
Assume that the VLSLS can be approximated by a small perturbation, \( \eta_0 \), superimposed on a vertical slowness, \( \gamma_0 \), in a laterally homogeneous embedding,

\[
\gamma(x_T, \alpha_T) \approx \gamma_0(\alpha_T) + \eta(x_T, \alpha_T).
\]

Compared with the dual-domain expression for wide-angle scattering from a thin slab in acoustic media (Wu and Huang, 1995, Eq.(18)), we find that

\[
\eta_{WH}^{WH}(\alpha_T - \alpha_T', \alpha_T) = \alpha_0 \frac{\alpha_0}{\gamma_0(\alpha_T)} \left[ \frac{1}{2} \mathcal{E}_\kappa(\alpha - \alpha') \right] - \alpha_0 \mathcal{E}_\kappa(\alpha - \alpha')
\]

where \( \alpha_0 = (\rho_0 \kappa_0)^{1/2} = 1/c_0 \) is the medium slowness of the embedding, and \( \gamma_0(\alpha_T) \) denotes the horizontal / 3D spatial Fourier transform. To arrive at the wide-angle-screen approximation, we have to assume that \( \alpha \cdot \alpha' \approx \alpha_0^2 \) and that \( \gamma_0(\alpha_T) \approx \gamma_0(\alpha_T) \). For media with constant density, the symbol reduces according to

\[
\eta_{WS}^{WH}(x_T, \alpha_T) = \alpha_0 \frac{\alpha_0}{\gamma_0(\alpha_T)} \mathcal{S}_\kappa(x_T),
\]
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Figure 1. Dispersion relations of wide-angle screen propagators in the high frequency limit. For reference, the exact h-f curves are shown in ticked solid lines; the solid lines represent the phase-screen approximation. The dotted lines represent the original wide-angle screens. The other curves show the modified wide-angle screens:
(a) The case of complex $\gamma_2 = \sqrt{\sigma_0^2 - (\xi \sigma_T)^2}$ with $\text{Re}(\xi) = 1$ and $\text{Im}(\xi) = 0.01$ for the long dashed line, 0.05 for the dashed line, and 0.1 for the dotted-dashed line.
(b) The case of complex $\gamma_2$ with $\xi = 1 + 0.1i$ for $\delta\sigma/\sigma_0 = \pm 25\%$.
(c) The case of stretched $\gamma_1$ and $\gamma_2$ with $b = 0.7$, $\xi = 0.08 + 0.9i$ for $\delta\sigma/\sigma_0 = 45\%$.
(d) The case of stretched $\gamma_1$ and $\gamma_2$ with $b = 1.6$, $\xi = 0.1 + 1.0i$ for $\delta\sigma/\sigma_0 = -45\%$. 
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with
\[ S_c(x_T) = \frac{1}{2} \frac{a(x_T)^2 - \alpha_0^2}{\alpha_0^2} \approx \frac{a(x_T) - \alpha_0}{\alpha_0} = \frac{\delta a(x_T)}{\alpha_0}. \]

Because of the separation of variables \( x_T \) and \( \alpha_T \), the propagator associated with \( \eta_{\text{WS}} \) can be implemented by a dual-domain technique using Fast Fourier Transforms.

The phase-screen representation of the VSLS yields a further approximation, viz.,
\[ \eta_{\text{WS}}(x_T, \alpha_T) \approx \eta_{\text{PS}}(x_T) = a_0 \, S_c(x_T), \quad (6) \]

Which leads to further algorithmic advantages. The VSLS is now decomposed into functions of space and of wave slowness.

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A comparison between the exact and screen propagators in the h-f limit seems controversial. For decreasing wave lengths, the h-f asymptotics becomes increasingly accurate, but the screen approximation may deteriorate. The screen approximation is particularly useful for lateral heterogeneities on the scale of the wave length. However, the geometry of the (bi)characteristics, and hence of the wave front, is fully determined by the h-f symbols. The wave shape can only be analyzed in particular lateral medium profiles.

Figure 1 shows the dispersion relations of screen propagators in the h-f limit compared with the leading term of the asymptotic expansion (principal part) of the VSLS, which is shown in ticked solid lines for reference. The dotted lines are the VSLS’s of the original wide-angle-screen approximation, and the solid lines are the ones of the phase-screen approximation. The horizontal axes are the radial horizontal slowness, and the vertical axis, the real part of vertical slowness. We conclude from Figure 1(a) that the wide-angle-screen propagator is more accurate than the phase-screen propagator but that the first propagator has a branch point at the embedding’s critical angle. This behavior is constrained to negative velocity perturbations. Around the branch point, the vertical slowness increases drastically, which induces an unphysical deformation of the wave front and can cause numerical dispersion and instability. In order to overcome this problem, we introduce a modification of the wide-angle VSLS:
\[ \gamma_{\text{WSC}}(x_T, \alpha_T) = \gamma_1(\alpha_T) + a_0 \frac{\alpha_0}{\gamma_2(\alpha_T)} \, S_c(x_T), \]
\[ \gamma_1(\alpha_T) = \sqrt{\alpha_0^2 - (b\alpha_T)^2}, \quad (7) \]
\[ \gamma_2(\alpha_T) = \sqrt{\alpha_0^2 - (\xi\alpha_T)^2}, \]

where \( b \) is a real number, and \( \xi \) is a complex number with a small imaginary part.

Figure 1(a) shows an example of complex \( \gamma_2 \); the perturbation is \( \delta a/a_0 = 20\% \). Not only is the branch point shifted into the complex plane, but the accuracy at large angles is improved as well. Figure 1(b) is the case of complex \( \gamma_2 \) with \( \xi = 1 + 0.1 i \); \( \delta a/a_0 = \pm 25\% \), which corresponds to a typical velocity contrast across a salt boundary from 4000 m/s (salt) to 2200 m/s (shale). For the case of the embedding’s slowness being equal to the median slowness, the dispersion relations of the complexified wide-angle propagator are shown as the dotted-dashed lines, for the scatterer (-25% perturbation) and for the host medium (+25% perturbation). The response of \( \gamma_{\text{WSC}} \) to the high velocity perturbation is similar to the one of \( \gamma_{\text{WS}} \), but the branch point of the negative velocity perturbation response has been shifted effectively. As shown later in the simulation of thin-slab transmission, the stability has been improved significantly. Figures 1(c) and (d) show the combined effects of wavenumber stretching and complexification. By optimizing the stretching and attenuating coefficients, the exact h-f dispersion relations can be matched very well for high velocity contrasts. However, the optimal match is strongly contrast dependent and hence is most useful for a single lateral medium transition.

To test the performance of screen propagators in laterally heterogeneous media with strong velocity contrasts, we use a thin slab with a ramp velocity profile as shown in the top panels of Figures 2 and 3. This ramp model could represent, again, the transition from salt to shales. In Figure 2 the velocity jumps from 2200 m/s to 3000 m/s, while in Figure 3 the velocity jumps from 2200 m/s to 4000 m/s. Plane waves illuminate the thin slab with different angles of incidence, \( \theta \); say. In Figure 2, the incident angles are \( 50^\circ \) and \( 34^\circ \) for the respective velocities. The time delays are calculated with the phase-screen propagator, \( \Delta \tau_{\text{PS}} \), with the wide-angle-screen propagator with real \( \gamma_2, \Delta \tau_{\text{WS}} \), and with the wide-angle-screen propagator with complex \( \gamma_2, \Delta \tau_{\text{WSC}} \). The dotted lines are the geometric optics predictions. We observe the superior accuracy of the wide-angle-screen propagators (the third and the fourth panels) over the phase-screen propagator (the second panel). Note also that the original wide-angle-screen propagator has a stability problem caused by the branch point, as shown by the oscillations on the curve. This instability is largely removed by complexifying \( \gamma_2 \). Figure 3 shows the case of a strong contrast and large incident angle (\( \theta = 65^\circ \)). The accuracy of the modified wide-angle-screen propagator (the fourth panel) is still satisfactory.

Conclusions

The phase-space path-integral formulation and the associated vertical slowness symbol provide a general and convenient framework for the accuracy analysis of screen propagators. From our analysis, we conclude that for weak medium perturbations both phase-screen and wide-angle-screen propagators perform well, while for larger contrasts a modification of the wide-angle-screen propagator is most appropriate. The modification yields a complexification as well as a stretch of the VSLS to
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Figure 2. Thin-Slab transmission tests using different screen propagators. The top panel is the velocity model. AT is the calculated time delay; PS refers to the phase-screen propagator, WS refers to the wide-angle-screen propagator with real $\gamma_2$, and WSC refers to the wide-angle-screen propagator with complex $\gamma_2$. The dotted lines are the geometric optics predictions. The velocity jumps from 2200 m/s to 3000 m/s ($\phi = 50^\circ$). It can be seen that the wide-angle-screen propagator with complex $\gamma_2$ generates the best results.

control the singularity and the artificial anisotropy due to the approximation.

Figure 3. As Figure 2. The velocity jumps from 2200 m/s to 4000 m/s ($\phi = 65^\circ$). It can be seen that the wide-angle-screen propagator with complex $\gamma_2$ generates the best results.

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