Scattering Characteristics of Elastic Waves by an Elastic Inclusion: I. Rayleigh Scattering of Elastic Waves

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Elastic wave scattering by a general elastic inclusion, having slightly different density and elastic constants from the surrounding medium, is formulated using body-force equivalent method and Born approximation. In low-frequency range (Rayleigh scattering), the scattered field by an arbitrary shaped inclusion having an arbitrary distribution of density and elastic constants can be equated to a radiation field from a point source composed of an unidirectional force proportional to the density contrast between the inclusion and the medium, and a force moment tensor proportional to the contrasts of the elastic constants. It is also shown that the scattered field can be decomposed into an "impedance-type", which has a main lobe in the backscattering direction and no scattering in the exact forward direction, and a "velocity-type" scattered field, which to the contrary has a main lobe in the forward scattering direction and no scattering in the exact backward direction. The resultant scattering pattern varies depending on various combinations of density and elastic-constant perturbations. Some examples of scattering pattern are given to show the general characteristics of elastic wave scattering.

Suppose the surrounding medium has a set of parameters \( \rho_0, \lambda_0, \mu_0 \), and the inclusion,

\[
\begin{align*}
\rho &= \rho_0 + \delta \rho, \\
\lambda &= \lambda_0 + \delta \lambda, \\
\mu &= \mu_0 + \delta \mu,
\end{align*}
\]

where the following conditions are supposed to be satisfied

\[
\delta \rho \ll \rho_0, \quad \delta \lambda \ll \lambda_0, \quad \delta \mu \ll \mu_0
\]

By using Born approximations, the scattered field can be calculated as the radiation field of the equivalent body forces due to the interaction of the primary wave and the inclusion. In Rayleigh scattering, since the size of the inclusion is small compared with the wavelength involved, phase differences between the radiation far fields of equivalent body force from different parts of the inclusion can be neglected. The total equivalent body force then can be regarded as a point force. The total equivalent single force and the total force moment tensor can therefore be derived by integrating over the volume of the inclusion. Knowing the equivalent point source, the scattered fields can be easily obtained by using elastodynamic Green’s function. For the plane P-wave incidence, we take the spherical coordinates having polar axis in the incident direction \( x_1 \) (i.e., in the direction of particle motion) (Figure 1). The scattered P-wave \( \mathbf{U}_P^s \) and S-wave \( \mathbf{U}_{S\text{inc}}^s \) can be written as

\[
\begin{align*}
\mathbf{U}_P^s &= \frac{V}{4\pi \omega^2} \cdot \left\{ \frac{\delta \rho}{\rho_0} \cos \theta - \frac{\delta \lambda}{\lambda_0 + 2\mu_0} + \frac{2\delta \mu}{\lambda_0 + 2\mu_0} \cos^2 \theta \right\} \frac{1}{r} e^{-i\omega(t - r/c_0)}, \\
\mathbf{U}_{S\text{inc}}^s &= \frac{V}{4\pi \beta_0^2} \cdot \left\{ -\frac{\delta \rho}{\rho_0} \sin \theta + \frac{\beta_0}{\omega} \frac{\delta \mu}{\mu_0} \sin 2\theta \right\} \frac{1}{r} e^{-i\omega(t - r/c_0)},
\end{align*}
\]

![Fig. 1. Spherical coordinate system for P-wave incidence and the scattering patterns for different equivalent forces.](image)

![Fig. 2. Scattering patterns of Rayleigh scattering for plane P-wave incidence. Upper half is of \( P-P \) scattering, the lower half \( P-S \) scattering. All patterns are axially symmetric about the x-axis.](image)
where $V$ is the volume of the inclusion, $\alpha_0$, $\beta_0$ are the $p$ and $s$-wave velocity in the medium respectively, subscript $r$ stands for $r$-component, and mer for meridian component. A bar over a parameter means taking average of it. Because of the symmetry of the problem with respect to the polar axis, there is no latitudinal component of $S$ wave, i.e., $\delta \mu V_{lat} = 0$.

Figure 1 shows the equivalent body forces and the corresponding scattering patterns of $pU_p^r$ and $pU_{mer}^r$ for $\delta \rho V$, $\delta \lambda V$, and $\delta \mu V$ under $P$-wave incidence. In general, if $\delta \rho/\alpha_0 = \delta \mu/\beta_0$ and $\lambda_0 = \mu_0$, we can decompose the scattered field into an impedance-type and a velocity-type scattered field,

$$pU_p^r = \frac{V}{4\pi \alpha_0^2} \frac{\omega^2}{r} e^{-i\omega t - r/r_0} \frac{\delta \rho}{\alpha_0} \left( \cos \theta - \frac{1}{3} \frac{\cos^2 \theta}{3} \right),$$

$$pU_{mer}^r = -\frac{V}{4\pi \beta_0^2} \frac{\omega^2}{r} e^{-i\omega t - r/r_0} \frac{\delta \rho}{\beta_0} \left( \sin \theta + \frac{\beta_0}{\alpha_0} \sin 2\theta \right),$$

where $Z_p$ and $Z_m$ are the $P$ and $S$-wave impedances, respectively.

Figure 2 shows the spatial patterns of the impedance type and the velocity type scattered field. Figure 3 gives two examples of the resultant scattering pattern.

For plane $S$-wave incidence, we take the direction of particle motion of the incident field ($y$-axis) as the polar axis of the spherical coordinates (Figure 4). The scattered $P$-wave $'U_p^r$ and the scattered $S$-wave $'U_{mer}^r$ and $'U_{lat}^r$ can be written as

$$'U_p^r = \frac{V}{4\pi \alpha_0^2} \frac{\omega^2}{r} \frac{1}{e^{-i\omega t - r/r_0}} \left( \delta \rho \frac{\sin \theta}{\alpha_0} \right),$$

and

$$'U_{mer}^r = -\frac{V}{4\pi \beta_0^2} \frac{\omega^2}{r} \frac{\delta \mu}{\mu_0} \cos \theta \cos \phi \frac{1}{e^{-i\omega t - r/r_0}},$$

where $\beta_0 = \mu_0$.

We plot the equivalent body forces and the corresponding scattering patterns in Figure 4. The resultant scattering patterns for Rayleigh scattering in $x$-$y$ plane for plane $S$-wave incidence. The upper half is of $S$-$S$ scattering, the lower half $S$-$P$ scattering.
For both \( P \) and \( S \)-wave incidences, converted waves (\( P-S \) or \( S-P \)) have only side lobes (with respect to the incident direction), while nonconverted scattered waves always have main lobes along the incident direction (either forward or backward direction). The impedance type scattered field will have no forward lobe, and the velocity type field will have no back lobe.

Figure 5 gives the scattering patterns for impedance type and velocity type scattered fields. Some examples of resultant scattering pattern are shown in Figure 6.

Beyond Rayleigh scattering, the equivalent forces of the scattered fields can no longer be regarded as a point source. The phase differences of the incident field at different parts of the inclusion and of the scattered field from different parts of the inclusion can no longer be ignored. Nevertheless, if the total scattered field is still much weaker than the incident field, the Born approximation can still be a useful tool for calculating the scattered field and deriving the scattering characteristics.

Suppose the density excess \( \delta \rho \), and the excesses of the elastic constants \( \delta \lambda \) and \( \delta \mu \) have the same spatial distribution within the inclusion. Using the Fraunhofer approximation, we derive the scattered far field for a finite volume elastic inclusion as follows

\[
\begin{align*}
U^P(x) &= \sum_{l=1}^{n} U^P_{l}(x) \Theta_l(x), \\
U^S(x) &= \sum_{l=1}^{n} U^S_{l}(x) \Theta_l(x), \\
U^\perp(x) &= \sum_{l=1}^{n} U^\perp_{l}(x) \Theta_l(x), \\
U^\parallel(x) &= \sum_{l=1}^{n} U^\parallel_{l}(x) \Theta_l(x),
\end{align*}
\]

Scattering Characteristics of Elastic Waves by an Elastic Inclusion: II. Elastic Wave Scattering beyond Rayleigh Scattering
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Beyond Rayleigh scattering, the equivalent forces of the scattered fields can no longer be regarded as a point source. It is shown in this paper that the scattered far-field can be obtained as a product of two factors. One is that of elastic wave Rayleigh scattering, the other is a scalar wave scattering factor for the finite body, which we call "volume interference factor," of which we derive here the explicit expressions for a spherical inclusion. The general scattering pattern will depend on different combinations of density and elastic-constant perturbations and also on the ratio of wavelength to the size of the inclusion. Some examples are given to show the general characteristics.

Beyond Rayleigh scattering, when the size of the inclusion becomes comparable to the wavelength, the equivalent forces of scattering by an inclusion can no longer be regarded as a point source. The phase differences of the incident field at different parts of the inclusion and of the scattered field from different parts of the inclusion can no longer be ignored. Nevertheless, if the total scattered field is still much weaker than the incident field, the Born approximation can still be a useful tool for calculating the scattered field and deriving the scattering characteristics.

Suppose the density excess \( \delta \rho \), and the excesses of the elastic constants \( \delta \lambda \) and \( \delta \mu \) have the same spatial distribution within the inclusion. Using the Fraunhofer approximation, we derive the scattered far field for a finite volume elastic inclusion as follows

\[
\begin{align*}
P^P(x) &= P^P_{l}(x) \Theta_l(x), \\
P^S(x) &= P^S_{l}(x) \Theta_l(x), \\
S^P(x) &= S^P_{l}(x) \Theta_l(x), \\
S^S(x) &= S^S_{l}(x) \Theta_l(x),
\end{align*}
\]