Renormalized scattering series for frequency domain waveform modelling of strong velocity contrasts

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**SUMMARY**

An improved description of scattering and inverse scattering processes in reflection seismology may be obtained on the basis of a scattering series solution to the Helmholtz equation, which allows one to separately model primary and multiple reflections. However, the popular scattering series of Born is of limited seismic modelling value, since it is only guaranteed to converge if the global contrast is relatively small. For frequency domain waveform modelling of realistic contrasts, some kind of renormalization may be required. The concept of renormalization is normally associated with quantum field theory, where it is absolutely essential for the treatment of infinities in connection with observable quantities. However, the renormalization program is also highly relevant for classical systems, especially when there are interaction effects that act across different length scales. In the scattering series of De Wolf, a renormalization of the Green functions is achieved by a split of the scattering potential operator into fore- and back-scattering parts; which leads to an effective reorganization and partially re-summation of the different terms in the Born series, so that their order better reflects the physics of reflection seismology. It has been demonstrated that the leading (single return) term in the De Wolf series (DWS) gives much more accurate results than the corresponding Born approximation, especially for models with high contrasts that lead to a large accumulation of phase changes in the forward direction. However, the higher-order terms in the DWS that are associated with internal multiples have not been studied numerically before. In this paper, we report from a systematic numerical investigation of the convergence properties of the DWS which is based on two new operator representations of the DWS. The first operator representation is relatively similar to the original scattering potential formulation, but more global and explicit in nature. The second representation is based on the T-operator formalism from quantum scattering theory, that offers a different perspective on the interaction between up- and down-going waves, as well as significant computational advantages (e.g., domain decomposition.
and fast recursive methods for one-way propagators). Our numerical results demonstrate the the convergence properties of the DWS are indeed superior to those of the Born series.
1 INTRODUCTION

Seismic forward modelling is an essential part of the interpretation or formal inversion of seismic data. Therefore, a huge effort has already been devoted to the development of different methods for seismic forward modelling (Carcione, 2002). However, there is still an important need for further development methods for seismic forward modelling, especially in connection with attempts to overcome the fundamental problem with convergence toward local minima within the context of full waveform inversion (e.g., Wu et al. 2014; Lesange et al., 2014; Jakobsen and Ursin, 2015). There exist a range of different methods for seismic forward modelling, including purely numerical methods (Carcione, 2002), asymptotic ray theory (Cerveny, 2011) and/or scattering theory (Weglein, 2003; Wu et al., 2007; Jakobsen, 2012). The different methods all have different features that can be regarded as an advantage or disadvantage, depending on the background of the researcher as well as the context and application. Asymptotic ray theory can be very efficient for smooth models that are large compared to the seismic wavelength (Cerveny, 2011), but more accurate results will generally be provided by a purely numerical method (Carcione, 2002) or scattering theory (Jakobsen, 2012), which can both be used for full waveform modelling. The fact that ray theory is event-oriented in the sense that one can model and identify particular ray and wave events (e.g., turning rays and primary or multiple reflections) is major advantage of this approach compared to the purely numerical methods. However, we can achieve more or less the same event-oriented advantage by using a scattering approach, which is both accurate and efficient compared to the purely numerical methods (Wu et al., 2007; Innanen, 2008, 2009).

Many geophysicists are familiar with the scattering series of Born that one can easily obtain from the Lippmann-Schwinger equation via iteration (e.g., Morse and Feshbach, 1953; Cohen and Bleistein, 1977, 1979; Clayton and Stolt, 1981; Jakobsen, 2012). However, the Born series represents an example of a so-called naive perturbation expansion which is only garantied to converge in the special case of small contrast volumes (Morse and Feshbach, Weinberg, 1963; Sams and Kouri, 1969a,b; Kirkinis, 2008; Jakobsen, 2012). In the presence of strong contrasts, it may be required to perform some kind of renormalization (De Wolf, 1971, 1985; Ostahev and Tatarski, 1995; Pankratov et al., 1995; Wu et al., 1996; Kouri et al., 2003; Innanen, 2009; Kirkinis, 2012; Jakobsen, 2012; Lesange et al. 2014). The term renormalization is often associated with quantum field theory and related mathematical structures (see Delamotte, 2004; Huang, 2013), but within the context of classical physics the term renormalization may refer to a split of the scattering potential operator into fore- and back-scattering parts, that effectively leads to a reorganization and partial resummation of the different scattering terms,
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so that their order better reflects the physics of wave propagation (De Wolf, 1985; Ostahev and Tatarski, 1995), for example related with reflection seismology (Wu et al., 1996; Innanen, 2008, 2009; Wu et al., 2014).

In this paper, we develop a renormalized scattering series method for frequency domain waveform modelling in the presence of large contrasts. By introducing several new scattering concepts into the scattering series formalism of De Wolf (see Wu et al., 2007), we obtain a more complete theory of multiple scattering in strongly scattering media that allows one to do individual, surgical forward modelling, first of the primaries, and then of the multiples of the reflected wavefield. The ability to turn multiples on and off during seismic forward modelling and geological hypothesis testing is one potential application of the work reported here (see Innanen, 2009). However, the main idea behind this study was to establish a more solid basis for the future development of direct nonlinear inversion methods. The renormalization procedure developed here was inspired by the work of Kouri et al. (2003), but our approach is different and not restricted to 1D media (see Lesage et al., 2014).

Wu et al. (2007) have developed efficient recursive (thin-slab) methods for one-way propagators and demonstrated that the single-return De Wolf approximation is superior to the corresponding Born approximation in the presence of strong contrasts. Wu and his co-workers have not performed a systematic numerical study of the higher-order terms in the De Wolf series that are associated with the internal multiples. Innanen (2009) have discussed the effects of internal multiples, but we believe this paper represents the first systematic numerical investigation of the convergence properties of the De Wolf series. The theory presented here can in principle be implemented using fast recursive (thin-slab) methods for one-way propagators (Wu et al., 2007). However, we have developed new operator representations of the De Wolf series that may be implemented using slightly different methods that are not only efficient, but allows for a more transparent and different perspective, that could be important for future theoretical developments. In what follows, we shall work in the frequency domain and employ the acoustic approximation for scalar media with variable wave speed and constant density. As discussed by Jakobsen and Ursin (2015), frequency domain methods can be very efficient for waveform inversion problems involving large numbers of sources, and also allows one to include attenuation effects via the use of complex-valued and frequency-dependent medium parameters. The acoustic approximation is commonly used for imaging inversion purpose, since it significantly reduces the computational cost. The work reported here may in principle be generalized to more realistic elastic media using a higher-order representation of the Lippmann-Schwinger equation (Jakobsen and Hudson, 2003; Jakobsen and Ursin, 2015; Ap-
pendix D); but we think it is convenient to use the acoustic approximation when introducing fundamentally new concepts and modelling methods.

In what follows, we shall first describe the naive and renormalized scattering series of Born and De Wolf. Then we shall discuss the De Wolf series method from a T-matrix perspective and derive several new results based on a general decomposition method for the T-operator. Finally, we shall discuss the results of several numerical experiments and provide some concluding remarks.

2 THE SCATTERING SERIES OF BORN

2.1 Lippmann-Schwinger equation for the Green function

The Green function for the scalar wave equation in the frequency domain (the Helmholtz equation) satisfies (Morse and Feshbach, 1953)

$$\left( \nabla^2 + \frac{\omega^2}{c^2(x)} \right) G(x, x') = -\delta(x - x'),$$

(1)

where \(c(x)\) is the wave speed at position \(x\) and \(\omega\) is the angular frequency. Defining \(c_0(x)\) as the wave speed in an arbitrary heterogeneous background medium, we get

$$\left( \nabla^2 + \frac{\omega^2}{c_0^2(x)} \right) G(x, x') = -\delta(x - x') - \omega^2 \chi(x) G(x, x'),$$

(2)

where

$$\chi(x) = c^{-2}(x) - c_0^{-2}(x)$$

(3)

is the perturbation of the squared slownesses. The second term on the right-hand side of equation (2) represents the so-called equivalent sources. By using the same source representation theorem for both real and (virtual) contrast-sources, we obtain the Lippmann-Schwinger equation (Morse and Feshbach, 1953; Jakobsen and Ursin, 2015)

$$G(x, x') = G^{(0)}(x, x') + \omega^2 \int_D \! dx'' G^{(0)}(x, x'') \chi(x'') G(x'', x'),$$

(4)

where \(D\) is the scattering domain where \(\chi(x'')\) is non-zero and \(G^{(0)}(x, x')\) is the background medium Green function, that satisfies

$$\left( \nabla^2 + \frac{\omega^2}{c_0^2(x)} \right) G(x, x') = -\delta(x - x').$$

(5)

If the background medium is homogeneous then one can use simple analytical expressions for \(G^{(0)}(x, x')\). If the background medium is not homogeneous then one can of course use ray theory or the finite difference method to estimate or compute the background medium Green function. In this study, however, we use the above equation in conjunction with the
T-matrix approach (discussed below) to relate homogeneous and heterogeneous background media Green functions (see also Jakobsen and Ursin, 2015). For compatibility with Dirac’s bracket notation for linear integral operators (Taylor, 1972), the Lippmann-Schwinger equation (4) can be rewritten exactly in the form of a product of continuous matrices (Jakobsen and Ursin, 2015):

$$G(x, x') = G^{(0)}(x, x') + \int_D dx_1 dx_2 G^{(0)}(x_1, x_2)V(x_1, x_2)G(x_2, x'),$$

(6)

where

$$V(x_1, x_2) = \omega^2 \chi(x_1) \delta(x_1 - x_2).$$

(7)

### 2.2 The Green operator and its Born series

In operator notation, the Lippmann-Schwinger equation (6) can be written as

$$G = G^{(0)} + G^{(0)}VG.$$  

(8)

The above equation (8) has the following exact formal solution:

$$G = (I - G^{(0)}V)^{-1}G^{(0)}$$

(9)

where $I$ is the identity operator.

The solution (9) is valid independently of the contrast volume, but it involves the inversion of a huge operator or matrix (in the coordinate representation), which can be very costly in the case of a realistic model. In principle, one could try to solve the Lippmann-Schwinger equation by iteration. This leads to the well-known Born series:

$$G = G^{(0)} + \sum_{p=1}^{\infty} \left(G^{(0)}V\right)^{p}$$

(10)

The Born series is very popular due to its simplicity. However, the Born series represents an example of a naive perturbation expansion (Kirkinis, 2008) which is only guaranteed to converge if the contrast volume is small (Taylor, 1972; Newton, 1992; Wu et al., 2007). This implies that the norm of the operator $G^{(0)}V$ must be smaller than unity (Jakobsen, 2012). Since the scattering potential $V$ is proportional to $\omega^2$, it is clear that the convergence properties of a given velocity model will be generally better at low than high frequencies.

### 2.3 Data and domain equations

The Green operators in section 2.2 are unrestricted in the sense that we have not specified their matrix elements in any particular representation. For applications to surface reflection seismology, it is convenient to introduce a set of restricted Green operators $G_{RS}$, $G_{VS}$, $G_{VV}$,
$G_{VR}$ that are characterized by the following matrix elements in the real-space coordinate representation (see Jakobsen and Ursin, 2015; Appendix A):

\begin{align}
<x_r | G_{RS} | x_s> &= G(x_r, x_s), \\
<x | G_{VS} | x_s> &= G(x, x_s), \\
<x | G_{VV} | x'> &= G(x, x'), \\
<x_r | G_{RV} | x> &= G(x_r, x),
\end{align}

(11)-(14)

Here $x_s$ and $x_r$ refers to positions at the source and receiver surface, respectively; whereas $x$ and $x'$ refers to different positions within the scattering volume. The matrix elements of the corresponding Green operators for the reference medium are given by similar equations. The matrix elements of the scattering potential operator $V$ in the real-space coordinate representation are given by

$$<x | V | x'> = \omega^2 \chi(x) \delta(x - x').$$

(15)

Thus, the scattering potential operator $V$ is local, although the formulation is non-local. A local operator may be represented by a diagonal matrix in the coordinate representation (see Appendix A), whereas a non-local operator will be represented by a non-diagonal matrix.

For seismic waveform modelling in the frequency domain using a scattering series method, one can use the following data equation for the surface observable quantities:

$$G_{RS} = G_{RS}^{(0)} + G_{RV}^{(0)} V G_{VS}$$

(16)

where

$$G_{VS} = G_{VS}^{(0)} + \sum_{p=1}^{\infty} \left(G_{VV}^{(0)} V\right)^p G_{VS}^{(0)},$$

(17)

is the Born series for $G_{VS}$.

3 THE SCATTERING SERIES OF DE WOLF

3.1 Renormalization via a split of the scattering potential operator

In order to derive the De Wolf series (Wu et al., 2007), we now assume that the total scattering potential $V$ operator can be decomposed as

$$V = V^{(f)} + V^{(b)},$$

(18)

where $V^{(f)}$ and $V^{(b)}$ are the parts of $V$ that are responsible for multiple scattering in the forward and backward directions, respectively. By substituting the above expression into the Lippmann-Schwinger equation (8), we obtain
\[ G = G^{(f)} + G^{(f)}V^{(b)}G. \]  

where the forward scattering renormalized Green operator \( G^{(f)} \) satisfies an integral equation of the Lippmann-Schwinger type

\[ G^{(f)} = G^{(0)} + G^{(0)}V^{(f)}G^{(f)}. \]  

The solutions of the integral equations (19) and (20) can be formally written as

\[ G = (I - G^{(f)}V^{(b)})^{-1}G^{(f)}. \]  

(21)

and

\[ G^{(f)} = (I - G^{(0)}V^{(f)})^{-1}G^{(0)}. \]  

(22)

respectively. By iterating on the integral equation (19), we obtain

\[ G = G^{(f)} + \sum_{p=1}^{\infty} \left( G^{(f)}V^{(b)} \right)^p G^{(f)}. \]  

(23)

Equation (23) represents a solution of the Helmholtz equation in the form of a De Wolf series for the Green operator. The scattering series expression (23) for the Green operator \( G \) is more convenient than the formal expression (21), since forward and backward could refer to both up- or down-going waves, depending on the experimental configuration. To derive more specific formula relevant for reflection seismology, we need evaluate the scattering potential and forward renormalized Green operators for up- and down-going waves.

### 3.2 Scattering potentials for up- and down-going waves

From the physical interpretation of the restricted Green operators as well as the \( V^{(u)} \) and \( V^{(d)} \) operators, it follows that

\[ V^{(u)} = P^{(u)}V, \]  

(24)

\[ V^{(d)} = P^{(d)}V, \]  

(25)

where the projection operators \( P^{(u)} \) and \( P^{(d)} \) for up- and down-going waves are defined by

\[ P^{(u)} = \left( G^{(0)}_{VV} \right)^{-1}G^{(0,u)}_{VV}, \]  

(26)

\[ P^{(d)} = \left( G^{(0)}_{VV} \right)^{-1}G^{(0,d)}_{VV}, \]  

(27)

and satisfies

\[ P^{(u)} + P^{(d)} = I. \]  

(28)

Here, we have introduced the modified Green operators \( G^{(0,u)} \) and \( G^{(0,d)} \) for the reference medium that annihilates down- and up-going waves, respectively. The operators \( G^{(0,u)}_{VV} \) and
$G_{VV}^{(0,d)}$ are defined by their matrix elements, which can be written in Dirac’s bra-ket notation as

\[ <x|G_{VV}^{(0,u)}|x'> = G^{(0)}(x,x')H(z-z'), \]  
\[ <x|G_{VV}^{(0,d)}|x'> = G^{(0)}(x,x')H(z'-z). \]

(29) (30)

Here, $z$ and $z'$ are the z-components of the vectors $x$ and $x'$ and $H(z-z')$ denotes the Heaviside step function, which is defined to be zero if its argument is negative and positive if the argument is positive or zero. Thus, the Green operators $G_{VV}^{(0,u)}$ and $G_{VV}^{(0,u)}$ are represented by upper and (nearly) lower triangular matrices in the real-space coordinate representation.

### 3.3 Renormalized Green operators for up- and down-going waves

Having determined the scattering potentials $V^{(u)}$ and $V^{(d)}$ for up- and down-going waves, the corresponding renormalized Green functions can be determined by solving the following integral equations

\[ G^{(u)}_{RV} = G^{(0)}_{RV} + G^{(0)}_{RV}V^{(u)}G^{(u)}_{VV}, \]  
\[ G^{(u)}_{VV} = G^{(0)}_{VV} + G^{(0)}_{VV}V^{(u)}G^{(u)}_{VV}, \]  
\[ G^{(d)}_{VV} = G^{(0)}_{VV} + G^{(0)}_{VV}V^{(d)}G^{(d)}_{VV}, \]  
\[ G^{(d)}_{VS} = G^{(0)}_{VS} + G^{(0)}_{VS}V^{(d)}G^{(d)}_{VS}. \]

(31) (32) (33) (34)

Note that the renormalized $G^{(u)}_{RV}$ operator is given in terms of $G^{(u)}_{VV}$, and the solutions to the other equations (32-34) can be written as

\[ G^{(u)}_{VV} = \Omega^{(u)} G^{(0)}_{VV}, \]  
\[ G^{(d)}_{VV} = \Omega^{(d)} G^{(0)}_{VV}, \]  
\[ G^{(d)}_{VS} = \Omega^{(d)} G^{(0)}_{VS}, \]

(35) (36) (37)

where the forward renormalized scattering operators $\Omega^{(u)}$ and $\Omega^{(d)}$ for up- and down-going waves are formally given by

\[ \Omega^{(u)} = (I - G^{(0)}_{VV}V^{(u)})^{-1}, \]  
\[ \Omega^{(d)} = (I - G^{(0)}_{VV}V^{(d)})^{-1}. \]

(38) (39)

By using the definitions of the $V^{(u)}$ and $V^{(d)}$ operators in equations (24) and (25) in conjunction with the above equations (38) and (39), we find that

\[ \Omega^{(u)} = (I - G^{(0,u)}_{VV}V)^{-1}, \]

(40)
\Omega^{(d)} = (I - G^{(0,d)}_{V,V} V)^{-1}. \tag{41}

A Neumann-expansion of the scattering operators \Omega^{(u)} and \Omega^{(d)} may have improved convergence properties due to the nearly triangular nature of the modified Green’s functions \(G^{(0,u)}\) and \(G^{(0,d)}\). However, we have only used the exact solutions in equations (40) and (41) in our numerical experiments. In any case, the thin-slab (one-way propagator) method of Wu et al. (2007) can be regarded as an efficient implementation of the above equations.
3.4 V-matrix representation of the De Wolf series

As stated earlier, the concepts of forward and backward scattering will obviously depend on the experimental configuration. For a surface seismic reflection experiment, it is clear that if we start with a down-going wave, then the first back-scattering will generate an up-going wave, the second-backscattering a down-going wave, and the third backscattering an up-going wave again. Thus, only terms that contains an odd number of backscattering operators will reach the surface. Since $G_{RS}^{(f)} = G_{RS}^{(d)} = 0$, it follows from equation (23) that

$$G_{RS} = G_{RS}^{(0)} + \left( \sum_{p=1}^{\infty} \prod_{q=1}^{p} G^{(f_q)} V^{(b_q)} \right) G_{VS}^{(d)}.$$  (42)

where $G^{(f_q)} = G_{VV}^{(u)}$ and $V^{b_q} = V^{(u)}$ if $q$ is an even number, $G^{(f_q)} = G_{VV}^{(u)}$ and $V^{b_q} = V^{(d)}$ if $q$ is an odd number larger than 1, and $G^{(f_1)} = G_{RV}^{(u)}$.

From equation (42), we obtain the following approximation

$$G_{RS} = G_{RS}^{(0)} + G_{RV}^{(u)} V^{(u)} G_{VS}^{(d)} + G_{RV}^{(u)} V^{(u)} G_{VV}^{(d)} V^{(d)} G_{VV}^{(u)} V^{(u)} G_{VS}^{(d)} \cdots.$$  (43)

These equations are clearly consistent with our intuitive understanding of the scattering processes involved in reflection seismology. However, we can derive an even simpler representation of the De Wolf-series and a deeper insight into the interaction between up- and down-going waves by using other methods from the rich interdisciplinary literature on scattering theory.

4 T-MATRIX PERSPECTIVE

4.1 The T-operator and it’s Born series

Following the quantum mechanical potential scattering approach (Taylor, 1972; Newton, 1992; Pike and Sabatier, 2002), we now introduce a transition operator or $T$-matrix, which is defined by (Taylor, 1972; Stolt and Jacobs, 1980-1981; Weglein et al., 1981; Weglein and Silivia, 1981; Cohen et al., 1989; Carvalho and Weglein, 1991; Newton, 1992; Jakobsen et al., 2003; Jakobsen and Hudson, 2003; Jakobsen, 2012; Jakobsen and Ursin, 2015)

$$VG = TG^{(0)}.$$  (44)

In terms of the $T$ operator, the data equation (16) can then be rewritten exactly as (detailed by Jakobsen and Ursin, 2015)

$$G_{RS} = G_{RS}^{(0)} + G_{RV}^{(0)} TG_{VS}^{(0)}.$$  (45)

From the Lippmann-Schwinger equation (8), the defining relation (44) and the fact that the background medium is arbitrary, it follows that (e.g., Taylor, 1972; Newton, 2002; Stolt and
Thus, the T-operator satisfies an integral equation of the Lippmann-Schwinger type, independent of the source-receiver configuration. Equation (46) has the following formal solution (e.g., Taylor, 1972)

\[
T = V (I - G^{(0)}_{VV} V)^{-1},
\]  

(47)

which corresponds to the inversion of a huge matrix in the real-space coordinate representation (Jakobsen, 2012; Jakobsen and Ursin, 2015; Levinson and Markel, 2015). In principle, \( T \) can also be evaluated using the following Neumann series:

\[
T = V + V \sum_{m=1}^{\infty} (G^{(0)}_{VV} V)^m.
\]  

(48)

The convergence properties of the above series are closely related to the convergence properties of the corresponding series for the Green operator in equation (10), which are relatively restricted.

### 4.2 General decomposition method for the \( T \)-operator

An arbitrary scattering domain with complete scattering potential \( V \) can always be decomposed into \( M \) (an arbitrary integer) components with scattering potentials \( V^{(m)} \), so that

\[
V = \sum_{m=1}^{N} V^{(m)}
\]  

(49)

If we assume that the corresponding T-operator can be written

\[
T = \sum_{m=1}^{M} T^{(m)}.
\]  

(50)

then it follows from the Lippmann-Schwinger equation (46) that relates the \( V \)- and \( T \)-operators that

\[
T^{(m)} = V^{(m)} + V^{(m)} G^{(0)}_{VV} \sum_{n=1}^{M} T^{(n)}.
\]  

(51)

The above equation (51) can be rewritten exactly as (Jakobsen, 2012)

\[
T^{(m)} = t^{(m)} + \sum_{n=1}^{M} t^{(m)} G^{(0)}_{VV} (1 - \delta_{mn}) T^{(n)},
\]  

(52)

where

\[
t^{(m)} = V^{(m)} (I - G^{(0)}_{VV} V^{(m)})^{-1}.
\]  

(53)
Equations (52-53) represents a separation of inter- and intra-component interaction of multiple scattering terms. By intra- and inter-component scattering, we mean scattering within a single domain or between different domains, respectively. The complete T-operator works on all space, but $t^{(m)}$ is a restricted (self-interaction or intra-domain scattering) operator that works only on the $m$th component of the scattering potential. In other words, computation of the small t-operators is characterized by a relatively small computational cost compared to the computation of the full T-operator.

### 4.3 Exact results for 2-component scattering potentials

The special case of 2-component scattering deserves special attention since the exact results that exists for this special case can be used in a recursive aggregate manner to study a general multi-layer system, and also to decompose the T-operator into interacting up- and down-going parts. For later reference, we assume now that the scattering potential operator $V$ can be decomposed as

$$V = V^{(1)} + V^{(2)}.$$  \hfill (54)

where $V^{(1)}$ and $V^{(2)}$ represents two independent components that depends on the context. The corresponding T-operator can be written as

$$T = T^{(1)} + T^{(2)}.$$  \hfill (55)

It follows from the general solution to the multi-component scattering problem in equation (52) that

$$T^{(1)} = t^{(1)} + t^{(1)}G_{VV}^{(0)}T^{(2)},$$  \hfill (56)

$$T^{(2)} = t^{(2)} + t^{(2)}G_{VV}^{(0)}T^{(1)},$$  \hfill (57)

where $t^{(1)}$ and $t^{(2)}$ are given by equation (53). For later reference, we note that the coupled integral equations (56-57) can be re-expressed exactly as

$$\left( I - t^{(1)}G_{VV}^{(0)}t^{(2)}G_{VV}^{(0)} \right)T^{(1)} = t^{(1)} + t^{(1)}G_{VV}^{(0)}t^{(2)}$$  \hfill (58)

$$\left( I - t^{(2)}G_{VV}^{(0)}t^{(1)}G_{VV}^{(0)} \right)T^{(2)} = t^{(2)} + t^{(2)}G_{VV}^{(0)}t^{(1)}.$$  \hfill (59)

The alternative form of the exact analytical results for 2-component scattering potentials given in equations (58-59) will be useful when developing a fast recursive aggregate method for construction of forward renormalized Green functions for up- and down-going waves.
4.4 Decomposition of the $T$ operator into interacting up- and down-going parts

If one decomposes the scattering potential $V$ into up- and down-going parts as shown in equation (18), then it follows from the exact results for 2-component scattering potentials in section 4.3 that the overall $T$-operator is given by (Appendix C)

$$ T = T^{(Ud)} + T^{(Du)} \tag{60} $$

where

$$ T^{(Ud)} = T^{(u)} + T^{(u)} G^{(0)}_{VV} T^{(Du)} \tag{61} $$

$$ T^{(Du)} = T^{(d)} + T^{(d)} G^{(0)}_{VV} T^{(Ud)} \tag{62} $$

and

$$ T^{(u)} = V^{(u)} \Omega^{(u)}, \tag{63} $$

$$ T^{(d)} = V^{(d)} \Omega^{(d)}. \tag{64} $$

Here, $\Omega^{(u)}$ and $\Omega^{(d)}$ are the scattering operators for up- and down-going waves given by equations (38-39). In equations (60-62), the $T^{(Ud)}$ and $T^{(Du)}$ operators represent the effects of interaction between up- and down-going waves on the operators $T^{(u)}$ and $T^{(d)}$, which, in turn, are associated with the forward renormalized propagators for up- and down-going waves, respectively.

Solving the coupled integral equations (61)-(62) via iteration, we obtain

$$ T^{(Ud)} = T^{(u)} + T^{(u)} \sum_{p=1}^{\infty} \prod_{q=1}^{p} G^{(0)}_{VV} T^{(M_q)}, \tag{65} $$

$$ T^{(Du)} = T^{(d)} + T^{(d)} \sum_{p=1}^{\infty} \prod_{q=1}^{p} G^{(0)}_{VV} T^{(M_{q+1})}, \tag{66} $$

where $M_q$ is equal to $d$ and $u$ when $q$ is an odd or even number, respectively. The above formula represents a renormalized scattering series for the overall $T$-operator that is different from the De Wolf series, but based on essentially the same philosophy; namely, a decomposition of the scattering operator into fore- and back-scattering parts.

The renormalized scattering series solution of the Lippmann-Schwinger equation (46) for the overall $T$-operator represented by equations (60) and (65-66) treat up- and down-going waves completely symmetric. This is in contrast to the $V$ and $T$-operator representations of the De Wolf series for the source-receiver Green operator $G_{RS}$ that is more closely connected with the experimental configuration. However, this feature is consistent with the fact that $T$-operator should be independent of the source-receiver configuration, and only represent the intrinsic scattering properties of the medium.
4.5 Recursive thin-slab method for the $T^{(u)}$ and $T^{(d)}$ operators

In this subsection, we shall first construct T-matrices for up- and down-going waves in a simple 2-layer system, and then discuss how to construct forward renormalized propagators for up- and down-going waves in a general (thin-slab) system by using the exact analytical results for a 2-layer system in subsection 4.3 in a recursive aggregate manner. The derivation presented here is completely novel, but has some conceptual similarities with the works of Wang and Chew (1992) as well as Wu et al. (2007).

For an up-going wave in a 2-layer system, we should include scattering-paths starting in the lower domain 2 and ending in the upper domain 1, but ignore scattering-paths that start in the upper domain 1 and ends in the lower domain 2. For a down-going wave in a 2-layer system, we should include scattering-paths starting in the upper domain 1 and ending in the lower domain 2. It follows from the decomposition of the T-operator in equation (55) and the exact relations in equations (58-59) that the $T^{(2,u)}$ and $T^{(2,d)}$ operators for up- and down-going waves in a 2-layer system are given exactly by

\[ T^{(2,u)} = t^{(1)} + t^{(2)} + t^{(1)} G^{(0)}_{VV} t^{(2)}, \]  
\[ T^{(2,d)} = t^{(1)} + t^{(2)} + t^{(2)} G^{(0)}_{VV} t^{(1)}, \]

respectively. We emphasize that $t^{(1)}$ and $t^{(2)}$ are the t-matrices for layers 1 and 2 when considered isolated. The last terms in equations (67) and (68) are associated with an interaction from layer 2 to layer 1 and from layer 1 to layer 2, respectively; so the above equations make good physical sense.

The results for a 2-layer system given in equations (67-68) can be used in a recursive manner to calculate the overall $T^{(u)}$- and $T^{(d)}$-operators for a general system with $N_3$ different layers or thin-slabs. We use the following algorithm:

\[ T^{(n,u)} = T^{(n-1,u)} + t^{(n)} + T^{(n-1,u)} G^{(0)}_{VV} t^{(n)}, \]  
\[ T^{(n,d)} = T^{(n-1,d)} + t^{(n)} + t^{(n)} G^{(0)}_{VV} T^{(n-1,d)}, \]

(69)  
\[ (70) \]

to compute the aggregate T-operators for up- and down-going waves at the $n$th recursion.

In the case of $N_3$ layers or thin-slabs, it is natural to think that the overall $T^{(u)}$ and $T^{(d)}$ operators are given by $T^{(N_3,u)}$ and $T^{(N_3,d)}$, respectively. However, it is required to include the projection operators $P^{(u)}$ and $P^{(d)}$ in the definition of the overall $T^{(u)}$ and $T^{(d)}$ operators;

\[ T^{(u)} = P^{(u)} T^{(N_3,u)}, \]  
\[ T^{(d)} = P^{(d)} T^{(N_3,d)}. \]  
\[ (71) \]  
\[ (72) \]
to ensure that we are indeed dealing with a one-way propagator. Recall that the projection operators $P^{(u)}$ and $P^{(d)}$ anihilates down- and up-going waves, respectively.
4.6 Renormalized Green operators

Having determined the $T^{(u)}$ and $T^{(d)}$ operators for up- and down-going waves, the corresponding forward renormalized Green operators (that are formally given by equations 20 and 44) can be computed using

\begin{align}
G^{(u)}_{RV} &= G^{(0)}_{RV} + G^{(0)}_{RV} T^{(u)} G^{(0)}_{VV}, \\
G^{(u)}_{VV} &= G^{(0)}_{VV} + G^{(0)}_{VV} T^{(u)} G^{(0)}_{VV}, \\
G^{(d)}_{VV} &= G^{(0)}_{VV} + G^{(0)}_{VV} T^{(d)} G^{(0)}_{VV}, \\
G^{(d)}_{VS} &= G^{(0)}_{VS} + G^{(0)}_{VV} T^{(d)} G^{(0)}_{VS},
\end{align}

As discussed earlier, the $T^{(u)}$ and $T^{(d)}$ operators can either be evaluated using the fast recursive scheme developed in section 4.5, or by noting that $T^{(u)} = \Omega^{(u)} V^{(u)}$ and $T^{(u)} = \Omega^{(u)} V^{(u)}$, where $\Omega^{(u)}$ and $\Omega^{(d)}$ are the corresponding scattering operators for up- and down-going waves given by equations (40-41).

4.7 T-matrix representation of the De Wolf series

By eliminating the scattering potential operators for up- and down-going waves and expressing all quantities in terms of the corresponding T-operators, one can re-express formula (42) for the source-receiver Green operator $G_{RS}$ exactly as

\begin{equation}
G_{RS}^{DWS} = \sum_{p=1}^{\infty} \prod_{q=1}^{p} G^{(0)}_{RV} T^{(M_q)} G^{(d)}_{VS},
\end{equation}

This is the T-matrix representation of the De Wolf series method for frequency domain waveform modelling in reflection seismology. If all terms in this renormalized scattering series are included then the results should be identical to those obtained using the exact T-operator, which can be obtained via the inversion of a huge operator or matrix. However, it is normally not required to include a large number of terms in the above series, since even the first term contains the most important higher-order effects associated with the accumulation of phase changes in the forward scattering direction. In order to analysis, the above equation numerically as well as analytically, it is convenient to write down the dominant terms explicitly.

From equation (77), we obtain the following approximation

\begin{equation}
G_{RS}^{DWS} = G^{(0)}_{RS} + G^{(0)}_{RV} T^{(u)} G^{(d)}_{VS} + G^{(0)}_{RV} T^{(u)} G^{(0)}_{VV} T^{(d)} G^{(0)}_{VV} T^{(d)} G^{(d)}_{VS} \cdots,
\end{equation}

The T-matrix representation of the DWS given above is equivalent to the V-representation in equation (43), but offers a simplified perspective and computational advantages (e.g., domain decomposition techniques and the fast recursive aggregate T-matrix method).
5 NUMERICAL EXAMPLES

5.1 Gaussian ball above a single reflector

We first consider a simple model of a Gaussian ball above a single horizontal reflector in conjunction with a homogeneous reference model (Figure 1). The scattering region including the Gaussian ball and the reflector, which is 510 m times 510 m in width and depth, is discretized into 51 times 51 grid blocks that are 10 m in each direction. We assume 51 receivers that are uniformly distributed along a single line at the top of the model, and we employ a single source located in the middle of the receiver line. The frequency used in this first numerical experiment is 10 Hz, corresponding to a wavelength which is significantly larger than the size of a single grid block, to avoid numerical discretization errors.

Figures 2 and 3 represents a comparison of the real and imaginary parts of the source-receiver Green function difference $\delta G_{RS} = (G_{RS} - G_{RS}^{(0)})$, which represents the scattered wavefield data in the frequency domain. The black and red curves are associated with the exact T-matrix and the Born series, respectively. Although the contrast volume is relatively high, the predictions of the first-order Born approximation are relatively good for the real part of the frequency domain waveform data, but not so good for the imaginary parts. However, the inclusion of higher-order terms in the Born series does not improve the results. In fact, one can clearly see that the Born series do not converge for this simple model characterized by a relatively high velocity contrast. The results in Figure 2 and 3 illustrate the fact that the Born approximation can be a relatively good approximation even when the Born series diverges. This kind of behaviour have been observed earlier in quantum scattering theory and is discussed in the book of Newton (1992).

Figures 4 and 5 are similar to Figures 2 and 3, but now we use the renormalized De Wolf series in place of the naive scattering series of Born. As expected, one can see that the (single return) De Wolf approximation is gives very accurate results for this simple model where multiples are not expected to play an important role.

5.2 Multiple curved layers

In order to illustrate the effects of the higher-order terms in the De Wolf series associated with internal multiples more clearly, we finally consider a strongly scattering medium where a Gaussian ball is squeezed in between multiple curved layers (Figure 6).

Figures 7 and 8 represents a comparison of the real and imaginary parts of the frequency domain waveform data we have constructed by using the exact T-matrix (black curves) and
the Born series (red curves) in the calculation of $\delta G_{RS}$. As expected, the Born series diverges for this strongly scattering medium, and the first-order Born approximation is useless.

Figures 9 and 10 are similar to Figures 12 and 13, but now we use the renormalized scattering series of De Wolf in place of the naive scattering series of Born. Clearly, one can see that the first-order De Wolf approximation that accounts for the primary reflections only have some errors, but the match between the exact T-matrix result (in black) the the De Wolf series predictions (in red) quickly becomes very good when we include the higher-order terms in the De Wolf series that are associated with internal multiples.

5.3 SEG/EAGE salt model

We then consider a more complex example of a strongly scattering medium; namely, the SEG/EAGE salt model (Figure 11). To reduce the computational cost down to a level which can be dealt with by a modern desktop computer, we resample the original model my using a moving averaging window that reduces the number of grid blocks with a factor of 4 in each direction. This means that we use 174 times 37 grid blocks in the vertical and horizontal directions, respectively. Since the each grid block is 10 m in each direction, the total size of the scattering domain is 1740 and 370 m in the horizontal and vertical directions, respectively.

Again, we employ a Ricker source wavelet with a central frequency equal to 15 Hz, so that the dominant wavelength is much larger than the size of a single grid block. We assume a single sources and 174 receivers that are uniformly distributed along a single source-receiver line at the top of the model.

Figures 12 and 13 represents a comparison of the real and imaginary parts of the frequency domain waveform data given by $\delta G_{RS}$ constructed using the exact T-matrix (black curves) and the Born series (red curves). Now, one can see that the first-order Born approximation is very inaccurate, and it does not help to include the higher-order terms in the Born series. In fact, one can clearly see that the Born series diverges for this strongly scattering medium.

Figures 14 and 15 are similar to Figures 12 and 13, but now we use the renormalized scattering series of De Wolf in place of the naive scattering series of Born. Clearly, one can see that the De Wolf series is stable and relievably accurate for this complex example of a strongly scattering medium. The first-order De Wolf approximation deviate slightly from the exact T-matrix result, and the match improves somewhat when we include the higher-order terms in the De Wolf series. However, it appears that it may be required to include higher-order terms beyond the 7-order internal multiples we have included in this numerical experiment. The fact that the salt body has a rough surface may explain why we did not obtain a perfect
match based on a 7-order De Wolf approximation. Despite this, we think the overall behaviour and results are very encouraging.
6 DISCUSSION

6.1 On the accuracy of the reference waveform modeling by the exact T-matrix method

In this study, we have used the exact T-matrix method to produce the reference waveforms required to numerically test the performance of various higher-order Born and T-matrix De Wolf series approximations in the frequency domain. One could of course have used another direct numerical simulation method (e.g., the finite difference method) to generate these reference waveforms. However, all the results and conclusions would have been the same, since the exact T-matrix method represents a full integral equation solution equivalent to the differential equation method (Jakobsen and Ursin, 2015).

For a 2D model similar to the curved layer model in Figure 6, Jakobsen and Ursin (2015) demonstrated that the predictions of the exact T-matrix method and an explicit finite difference time domain method based on a 9-point representation of the Laplacian operator produces very similar waveform modeling results in both time and frequency domains. For the more complicated SEG/EAGE salt model in Figure 11, we have arrived at the same conclusion in this study.

Figure 16 represents a comparison of time domain waveform modeling results obtained using the FDTD and exact T-matrix methods on the SEG/EAGE salt model in Figure 11. The calculation involved 174 receivers, but we show only the results for every 5th receiver. Clearly, one can see that the waveforms produced by the FDTD method (upper figure) and the exact T-matrix methods (middle figure) looks very similar. When we calculate the difference between the traces produced by the FDTD and exact T-matrix methods (lower figure), we see that the differences are indeed extremely small.

Figure 17 represents a comparison of frequency domain waveform modeling results obtained using the FDTD and exact T-matrix methods on the same SEG/EAGE salt model as in Figure 16. Here, we have organized the Fourier amplitudes for all 174 receivers at 3 Hz, 7.5 Hz and 15 Hz in the form of a data vector (typically used for frequency domain full waveform inversion). Again, one can see that the agreement between the predictions of the FDTD and exact T-matrix results is very good. One can see some tiny differences between the predictions by the exact T-matrix and the FDTD method at the highest frequency, but these differences are extremely small compared to the differences one can observe between the various higher-order Born and De Wolf series approximations in Figures 2-5 (for the Gaussian
ball model), 7-10 (for the curved layer model) and Figures 12-15 (for the SEG/EAGE salt model).

The fact that we have performed all the numerical experiments on the performance of the (renormalized) scattering series in the frequency domain, rather than in the time domain, should also make good sense; since our Green function and T-matrix approach to waveform modeling and inversion is also formulated in the frequency domain.

6.2 Scattering series methods vs full waveform methods

One reviewer asked why anyone who has access to numerical modeling methods such as finite difference, finite element, lattice Boltzmann etc would want to consider a scattering series solution, renormalized or otherwise. Although we have already provided several arguments in the introduction section, we think this question is so interesting that we shall now discuss the various arguments in a little more detail.

A numerical method typically provides the full waveforms, whereas a scattering series method allows one to separately model the primary reflections and internal (or surface-related) multiples of different order and also model the full waveforms (if one adds enough scattering terms). This ability to decompose the waveforms into specific terms that correspond to different events is very useful when testing conventional methods for seismic imaging inversion that only makes use of the primary reflections (see Moser, 2011), as well as when developing new methods for seismic imaging with internal multiples that may lead to improved illumination and imaging of sub-salt structures and related media (see Malcholm et al., 2005).

A purely numerical (black box) approach to seismic forward modelling is not suitable for the development of the more direct inversion methods promoted by Weglein et al. (2007); but we think that the renormalized scattering series derived in this study can be used to improve on the convergence properties of the inverse scattering series method. (see Kouri et al., 2003).

Our renormalized scattering series are similar in spirit to the Volterra structures of Kouri et al. (2003), but the De Wolf series has a very transparent physical interpretation and can be implemented in an extremely efficient manner using the thin-slab propagator method introduced Wu et al. (2007). The T-matrix perspective adds a new dimension to the previous work of Wu et al., since it allows for a different implementation (e.g., involving domain decomposition) as well as theoretical interpretation.

Finally, we would like to emphasize that this paper essentially represents a scattering theoretical study aimed at obtaining an improved insight into different wave phenomena that are relevant for modelling and inversion in strongly scattering media, rather than to develop
yet-another modelling method. We have made significant progress on the renormalization of the forward scattering series, but this does not necessarily imply a proper renormalization of the inverse scattering series (see Kouri et al., 2003).
7 CONCLUSIONS

We have developed two new operator representations of the renormalized scattering series method of De Wolf that can be used for frequency domain waveform modelling in the presence of strong contrasts. The first operator representation is relatively similar to an existing scattering potential operator representation in the time domain, but more global and explicit in nature. The second operator representation is based on the T-matrix approach of quantum scattering theory, which offers a different perspective on the interaction between up- and down-going waves, as well as significant computational advantages (e.g., domain decomposition and fast recursive methods for one-way propagators).

The work reported here represents the historic first systematic numerical study of the higher-order terms in the De Wolf series that are associated with internal multiples. We have confirmed that the leading (single return) De Wolf approximation is much more accurate than the corresponding Born approximation, and we have demonstrated that the convergence properties of the De Wolf series are indeed superior to those of the Born series.

The renormalized scattering series we have developed in this paper can be regarded as an interesting alternative to the finite difference or element methods for seismic forward modelling, which allows one to separate the effects of primary and multiple reflections. Also, the work reported here may be regarded as the first step towards a more direct nonlinear inversion method.

The fact that the renormalized Green operators for up- and down-going waves depends on the scattering potential operator, represents a major challenge for the development of direct nonlinear inversion methods (see also Kouri et al., 2003). However, the renormalized scattering series of De Wolf developed in this paper can easily be combined with the direct iterative T-matrix inversion methods introduced by Jakobsen and Ursin (2015). In any case, we think that the renormalized scattering series developed in this study represents a solid basis for future work within nonlinear inverse scattering in elastic as well as acoustic media.

8 ACKNOWLEDGMENTS

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APPENDIX A: REPRESENTATION OF INTEGRAL OPERATORS BY MATRICES

We use Dirac’s bra-ket notation to define the matrix elements of the restricted Green operators $G_{RS}$, $G_{VS}$, $G_{VV}$ and $G_{RV}$ in equations (11-14), the scattering potential operator $V$ in equation (15) and the upper and lower triangular parts $G_{VV}^{(u,0)}$ and $G_{VV}^{(d,0)}$ of the $G_{VV}$ operator in equations (29-30). Dirac’s bra-ket notation for infinite dimensional vectors in dual Hilbert spaces (see Taylor, 1972) is standard in quantum scattering theory and very convenient since it allows the multiple scattering theory to be formulated in a manner which is independent of the way one represents the operators. However, it is of course necessary to choose a particular integral operator representation when implementing the theory on a computer. Many researchers within the inverse scattering community seems to prefer the momentum (or Fourier) representation, which leads to higher-dimensional integrals (e.g., Pike and Sabatier, 2002). However, Jakobsen (2012) and Jakobsen and Ursin (2015) have recently developed a real space coordinate representation of the integral operators which means that all operators can be represented by matrices (see also Levinson and Markel, 2014). In this appendix, we provide a brief review of the real-space coordinate representation detailed by Jakobsen and Ursin (2015) and add some additional details related to the representation of the $V^{(u)}$ and $V^{(d)}$ operators.

Following Jakobsen (2012) and Jakobsen and Ursin (2015), we assume that there are multiple receivers located at positions $x_r$, where $r = 1, ..., N_r$; respectively. We divide the scattering domain $D$ where the scattering potential $V$ is non-zero into a set of $N$ grid blocks with centroid $x_p$ and volume $\delta v_p$, labelled by $p = 1, ..., N$. The size of the individual grid blocks should be chosen small compared to the dominant wavelength. We also introduce an index $n$ which may be associated with the field at a particular receiver position or inside a particular grid block within the discretized scattering volume. Discrete versions of the (symmetrized) Lippmann-Schwinger equation (6) can then be written as

$$G_{mn} = G_{mn}^{(0)} + \sum_{p=1}^{N} \delta v_p \sum_{q=1}^{N} \delta v_q G_{mp}^{(0)} V_{pq} G_{qn},$$

(A.1)

and (see Sheng, 1990)

$$V_{pq} = \omega^2 \chi(x_p) \frac{\delta_{pq}}{\delta v_p}.$$  

(A.2)

Here, $\delta_{pq}$ is the Kronecker-delta, defined by $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \neq q$.

For the discretization of Green function, one can use (Jakobsen, 2012; Jakobsen and Ursin, 2012; 2015; Levinson and Markel, 2014)
\( G_{pq}^{(0)} = G^{(0)}(x_p, x_q), \quad p \neq q, \)  \hfill (A.3)

and

\[
\delta v_p G_{pp}^{(0)} = \int_{D_p} d x G^{(0)}(x_p, x).
\]  \hfill (A.4)

Here, the symbol \( D_p \) denotes the domain occupied by a single grid block centred at position \( x_p \).

The calculation of the above integral of the Green function over a single grid block (related to the interaction of a single grid block with itself) is discussed in Jakobsen (2012). The singularity of Green function have effectively been dealt with by replacing a cubic grid block with an equivalent spherical domain with the same volume. Jakobsen (2012) integrated out the singularity of the 3D Green functions, but similar expressions for the 2D case can be found in the paper of Cao and Torres-Verdin (2005); which also includes non-singular integrals for improving on the off-diagonal terms of the Green matrix in equation (21). Jakobsen and Ursin (2015) verified that these corrections improves the match between T-matrix and finite difference simulation results.

From equations (A.1-A.2) it follows that

\[
G_{mn} = G_{mn}^{(0)} + \sum_{p=1}^{N} \sum_{q=1}^{N} \chi(x_p) \delta v_q \delta_{pq} G_{mp} V_{pq} G_{qn},
\]  \hfill (A.5)

where

\[
V_{pq} = \omega^2 \chi(x_p) \delta v_q \delta_{pq}.
\]  \hfill (A.6)

Equations (A.5-A.6) in conjunction with the definitions of the contrast function \( \chi(x) \) in equation (3) form the basis for the simple (coordinate) matrix representation we have used of all integral operators in this study. In matrix notation, equation (A.5) becomes

\[
G = G^{(0)} + G^{(0)} V G,
\]  \hfill (A.7)

which is identical to the operator equation (8). Our notation does not distinguish between an operator equation and it’s matrix representation. Following Jakobsen and Ursin (2015), however, we denote by \( G_{VS}, G_{VV}, G_{RV} \) and \( G_{RS} \) the Green operators or matrices that are restricted to have one or two arguments associated with the source-receiver surface or the scattering domain.

Again, we emphasize that all operators discussed in this paper can be represented by matrices using the real-space coordinate representation discussed above. This implies that all operator products and operator inverses can be replaced by matrix products and matrix inverses, respectively. Equations (24-25) shows that the \( V^{(u)} \) and \( V^{(d)} \) matrix (or operator) can be determined by operating on the scattering potential matrix \( V \) with the projection
matrices $P^{(u)}$ and $P^{(d)}$, which in turn are given by the (matrix) equations (26) and (27). With our definition of the Heaviside function, $G^{(d,0)}_{VV}$ indeed becomes the lower triangular part of the $G^{(0)}_{VV}$ matrix, whereas $G^{(u,0)}_{VV}$ is actually equal to the sum of the upper triangular and the diagonal part of $G^{(0)}_{VV}$. The results of several numerical experiments suggest that this choice leads to more accurate results than a decomposition of the scattering potential operator into three different terms; that is, a separate treatment of the diagonal terms of $G^{(0)}_{VV}$. 
APPENDIX B: DETAILED DERIVATION OF EQUATION (46)

We start by noting that the matrix of source-receiver Green functions $G_{RS}$ is given by

$$G_{RS} = G_{RS}^{(0)} + G_{RV}^{(0)} V G_{VS},$$  \hspace{1cm} (B.1)

where $G_{VS}$ is a matrix of source-volume Green functions that satisfy the Lippmann-Schwinger equation;

$$G_{VS} = G_{VS}^{(0)} + G_{VV}^{(0)} V G_{VS}. \hspace{1cm} (B.2)$$

Equations (B.1) and (B.2) may be referred to as the data and domain equations, respectively.

The main challenge here is to solve the Lippmann-Schwinger equation (B.2) for $G_{VS}$, so that the result can be substituted into the data equation (B.1).

The definition of the $T$-operator in equation (44) implies that

$$V G_{VS} = T G_{VS}^{(0)}. \hspace{1cm} (B.3)$$

Combining equations (B.2) and (B.3), we obtain

$$G_{VS} = G_{VS}^{(0)} + G_{VV}^{(0)} T G_{VS}^{(0)}. \hspace{1cm} (B.4)$$

Operating (or multiplying) equation (B.4) with $V$ from the left results in

$$V G_{VS} = V G_{VS}^{(0)} + V G_{VV}^{(0)} T G_{VS}^{(0)}. \hspace{1cm} (B.5)$$

It now follows from equations (B.3) and (B.5) that

$$T G_{VS}^{(0)} = V G_{VS}^{(0)} + V G_{VV}^{(0)} T G_{VS}^{(0)}. \hspace{1cm} (B.6)$$

The above equation can be rewritten exactly as

$$\left( T - V - V G_{VV}^{(0)} T \right) G_{VS}^{(0)} = 0. \hspace{1cm} (B.7)$$

The Green operator $G_{VS}^{(0)}$ depends on the reference medium. Since the reference medium can be selected arbitrarily, the Green operator $G_{VS}^{(0)}$ can also be regarded as arbitrary. Therefore, the factor inside the parentheses on the right-hand side of equation (B.7) must be identical to zero;

$$T - V - V G_{VV}^{(0)} T = 0. \hspace{1cm} (B.8)$$

It now follows from equation (B.8) that

$$T = V + V G_{VV}^{(0)} T. \hspace{1cm} (B.9)$$

Thus, we have now provided a detailed derivation of equation (46). We refer to the textbook of Newton (1992) for an even more detailed discussion of the T-matrix.
APPENDIX C: DETAILED DERIVATION OF EQUATIONS (60-64)

We assume first that the scattering potential operator $V$ can be decomposed as

$$V = V^{(u)} + V^{(d)},$$

where $V^{(u)}$ and $V^{(d)}$ represents the parts of the scattering potential $V$ that are responsible for up- and down-going waves, respectively. The corresponding $T$ operator can be written as

$$T = T^{(Ud)} + T^{(Du)},$$

It follows from the solution to the two-component scattering problem in equations (54-57) that

$$T^{(Ud)} = T^{(u)} + T^{(u)}G^{(0)}_{VV}T^{(Du)}$$

and

$$T^{(Du)} = T^{(d)} + T^{(d)}G^{(0)}_{VV}T^{(Ud)}$$

where

$$T^{(u)} = V^{(u)} \left( I - G^{(0)}_{VV}V^{(u)} \right)^{-1},$$

$$T^{(d)} = V^{(d)} \left( I - G^{(0)}_{VV}V^{(d)} \right)^{-1},$$

follows from the general equation (53). Thus, equations (60-64) follows directly from the exact results for 2-component scattering potentials in equations (54-57), if one makes the following substitutions:

$$V^{(1)} \rightarrow V^{(u)},$$

$$V^{(2)} \rightarrow V^{(d)},$$

$$T^{(1)} \rightarrow T^{(Ud)},$$

$$T^{(2)} \rightarrow T^{(Du)},$$

$$t^{(1)} \rightarrow T^{(u)},$$

$$t^{(2)} \rightarrow T^{(d)}.$$
APPENDIX D: GENERALIZATION TO ANISOTROPIC VISCO-ELASTIC MEDIA

In this study, we have employed an acoustic approximation to simplify the mathematical derivations and to reduce the computational cost. An acoustic approximation may lead to sufficiently accurate results for many seismic imaging problems, and also for some qualitative reservoir characterization and monitoring problems. However, a detailed characterization of petroleum reservoirs and other complex geological structures may require the use of a more general T-matrix De Wolf series that can account for elasticity, anisotropy and possibly even attenuation. The paper of Jakobsen and Ursin (2015) contains an appendix which discuss how one can generalize the T-matrix approach to anisotropic visco-elastic media. Jakobsen and Ursin (2015) shows that the velocity-strain field satisfies an integral equation of the Lippmann-Schwinger type, but in a 11-dimensional space. This implies that the all the ideas and methods developed in this study on the basis of the scalar wave equation can in principle also be generalized to the elastic wave equation, although the technical details may quickly become challenging.

Following Jakobsen and Hudson (2003), Jakobsen and Ursin (2015) and Jakobsen et al. (2015), we start by combining the stress tensor and the momentum density vector at position \( x \) into a (time-reduced) a 11-dimensional stress-momentum vector \( \Phi(x) \), defined by

\[
\Phi(x) \equiv [p(x), \sigma(x)]^T.
\]  

(D.1)

Since there are generally 3 and 6 independent momentum density and stress components, respectively, \( \Phi(x) \) is clearly a 11-dimensional vector. Similarly, we combine the strain tensor \( \epsilon(x) \) and the velocity vector \( i\omega u(x) \) into a strain-velocity vector \( \Psi(x) \), defined by

\[
\Psi(x) \equiv [i\omega u(x), \epsilon(x)]^T.
\]  

(D.2)

Again, there are generally 3 and 6 independent particle displacement and strain components, so this 11-dimensional formulation is clearly internally consistent.

From the discussion in the paper of Jakobsen and Hudson (2003), it is clear that the above vectors are related by the following

\[
\Phi(x) = R(x)\Psi(x),
\]  

(D.3)

where

\[
R(x) \equiv \begin{bmatrix}
\rho(x)I_2 & 0 \\
0 & C(x)
\end{bmatrix}.
\]  

(D.4)

Since \( C(x) \) is the local (frequency-dependent and complex-valued stiffness tensor and \( \rho(x) \) is
the corresponding mass density, it clearly makes sense to refer to \( R(\mathbf{x}) \) as the local stiffness-density matrix. The components of the second-rank identity tensor \( I_2 \) are given by \( (I_2)_{ij} = \delta_{ij} \), where \( \delta_{ij} \) is the symbol of Kronecher.

In order to make contact with the scalar formulation, we first decompose the stiffness-density matrix as

\[
R(\mathbf{x}) = R(\mathbf{x})^{(0)} + \delta R(\mathbf{x}),
\]

where

\[
\delta R(\mathbf{x}) = \begin{bmatrix}
\delta C(\mathbf{x}) & 0 \\
0 & \delta \rho(\mathbf{x}) I_2
\end{bmatrix},
\]

is the (possibly large) fluctuation of \( R(\mathbf{x}) \) from a quantity

\[
R(\mathbf{x})^{(0)} = \begin{bmatrix}
C(\mathbf{x})^{(0)} & 0 \\
0 & \rho(\mathbf{x})^{(0)} I_2
\end{bmatrix},
\]

which may be non-uniform in space. The choice of \( R(\mathbf{x})^{(0)} \) is arbitrary, but it may be convenient to choose a relatively simple background medium, depending on the application.

A higher-dimensional domain integral equation of the Lippmann-Schwinger type the reader may be familiar with from both seismics and quantum scattering theory can now be derived from the above equations and the equation of motion (see Jakobsen and Hudson, 2003; Jakobsen and Ursin, 2015) by using the concept of Green’s function;

\[
\Psi(\mathbf{x}) = \Psi(\mathbf{x})^{(0)} + \int_D d\mathbf{x}' G(\mathbf{x}, \mathbf{x}') \delta R(\mathbf{x}') \Psi(\mathbf{x}').
\]

Here \( \Psi(\mathbf{x})^{(0)} \) is the strain-velocity vector associated with wave motion in the background medium with properties given by \( R(\mathbf{x})^{(0)} \), and the second term on the right-hand side of equation (D.8) represents the wavefield scattered from the perturbations (or contrast-sources) \( \delta R(\mathbf{x}') \). In equation (D.8), we have introduced the generalized Green’s function \( G(\mathbf{x}, \mathbf{x}') \) for the background medium, which satisfies

\[
D R(\mathbf{x})^{(0)} G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') I_2.
\]

The solution to equation (D.9) is given by (Jakobsen and Hudson, 2003)

\[
G(\mathbf{x}, \mathbf{x}') = \begin{bmatrix}
S_t(\mathbf{x}, \mathbf{x}') & M_t(\mathbf{x}, \mathbf{x}') \\
S_x(\mathbf{x}, \mathbf{x}') & M_x(\mathbf{x}, \mathbf{x}')
\end{bmatrix},
\]

Here, the third- and fourth-rank tensors \( M_t(\mathbf{x}) \) and \( S_x(\mathbf{x}) \) are determined via single and double spatial differentiation of the elastic displacement Green’s function; whereas the second-rank rank tensors \( S_t(\mathbf{x}) \) and \( M_t(\mathbf{x}) \) are determined via single and double temporal differentiation (multiplication with different powers of the angular frequency \( \omega \), as discussed by Jakobsen.
and Hudson (2003). In operator notation, the 11-dimensional Lippmann-Schwinger equation (D.8) can be written as

\[ \Psi = \Psi^{(0)} + G^{(0)} V \Psi, \]  

(D.11)

In order to derive a De Wolf series for \( \Psi \), we next assume that the total scattering potential operator \( V \) can be decomposed as

\[ V = V^{(f)} + V^{(b)}, \]  

(D.12)

where \( V^{(f)} \) and \( V^{(b)} \) are the parts of \( V \) that are responsible for forward and backward scattering, respectively. By substituting the above equation into the Lippmann-Schwinger equation (10), we obtain

\[ \Psi = \Psi^{(f)} + G^{(f)} V^{(b)} \Psi, \]  

(D.13)

where

\[ G^{(f)} = G^{(0)} + G^{(0)} V^{(f)} G^{(f)}, \]  

(D.14)

and

\[ \Psi^{(f)} = G^{(f)} S, \]  

(D.15)

where \( S \) represents the source wavelet in the frequency domain. By iterating on the above equation, we obtain

\[ \Psi = \Psi^{(f)} + \sum_{p=1}^{N} \left( G^{(f)} V^{(b)} \right)^{p} \Psi^{(f)}. \]  

(D.16)

Alternatively, we can write

\[ \Psi = \Psi^{(f)} + G^{(f)} T^{(b)} \Psi^{(f)}, \]  

(D.17)

where the T-matrix is given by

\[ T^{(b)} = V^{(b)} + V^{(b)} G^{(f)} V^{(b)} + V^{(b)} G^{(f)} V^{(b)} G^{(f)} V^{(b)} + \cdots. \]  

(D.18)

The above equations just indicate how the T-matrix De Wolf series method that we have developed and implemented for scalar media can be generalized to anisotropic visco-elastic media. A more complete derivation that may include numerical examples for elastic media will be reported elsewhere. The main take home message here is that we can develop a T-matrix De Wolf series methods for anisotropic visco-elastic media if one employs the 11-dimensional Lippmann-Schwinger equation for the strain-velocity field.
REFERENCES


Weglein, A.B., 2013. A timely and necessary antidote to indirect methods and so-called P-wave FWI The Leading Edge October, 1192-1204.


Figure 1. A simple velocity model with only primary reflections are expected to be important. Displayed is a Gaussian ball above a single horizontal reflector.
Figure 2. Frequency domain waveform data for the Gaussian ball model in Figure 1 obtained using the naive scattering series of Born. Displayed are the real parts of the Fourier components at 10 Hz for a single line of 51 receivers and a single source in the middle of the receiver line.
Figure 3. The same as in Figure 3 but for the imaginary part of the frequency domain seismic data.
Figure 4. Frequency domain waveform data for the Gaussian ball model in Figure 1 obtained using the renormalized scattering series of De Wolf. Displayed are the real parts of the Fourier components at 10 Hz for a single line of 51 receivers and a single source in the middle of the receiver line.
Figure 5. The same as in Figure 5 but for the imaginary part of the frequency domain data.
Figure 6. A model with multiple curved interfaces where internal multiples may potentially play a more significant role.
Figure 7. Frequency domain waveform data for the model with curved layers in Figure 1 obtained using the naive scattering series of Born. Displayed are the real parts of the Fourier components at 10 Hz for a single line of 101 receivers and a single source in the middle of the receiver line.
Figure 8. The same as in Figure 8 but for the imaginary part of the frequency domain data.
Figure 9. Frequency domain waveform data for the model with curved layers in Figure 1 obtained using the renormalized scattering series of De Wolf. Displayed are the real parts of the Fourier components at 10 Hz for a single line of 101 receivers and a single source in the middle of the receiver line.
Figure 10. The same as in Figure 10 but for the imaginary part of the frequency domain data.
Figure 11. A resampled version of the SEG/EAGE salt model.
Figure 12. Frequency domain waveform data for the model with curved layers in Figure 1 obtained using the naive scattering series of Born. Displayed are the real parts of the Fourier components at 10 Hz for a single line of 101 receivers and a single source in the middle of the receiver line.
Figure 13. The same as in Figure 8 but for the imaginary part of the frequency domain data.
Figure 14. Frequency domain waveform data for the model with curved layers in Figure 1 obtained using the renormalized scattering series of De Wolf. Displayed are the real parts of the Fourier components at 10 Hz for a single line of 101 receivers and a single source in the middle of the receiver line.
Figure 15. The same as in Figure 10 but for the imaginary part of the frequency domain data.
Figure 16. Seismic waveforms in the time domain generated using the finite difference time domain (FDTD) method and the T-matrix approach for the SEG/EAGE salt model in Figure 6 with a Ricker wavelet source having a central frequency equal to 7.5 Hz.
Figure 17. Comparison of frequency domain waveform results generated using the FDTD method (red) and the T-matrix approach (blue). Data vector components 1-174, 175-349 and 350-522 corresponds to 3 Hz, 7.5 Hz and 15 Hz, respectively.