Wave equation least square imaging using the local angular Hessian for amplitude correction

Haoran Ren\textsuperscript{1,2}\textsuperscript{*}, Ru-Shan Wu\textsuperscript{2} and Huazhong Wang\textsuperscript{1}

\textsuperscript{1}School of Ocean and Earth Science, Tongji University, Siping Rd. 1239 Shanghai 200092, China, and \textsuperscript{2}Department of Earth and Planetary Sciences, Institute of Geophysics and Planetary Physics, University of California, Santa Cruz, CA 95064, USA

Received March 2010, revision accepted December 2010

ABSTRACT

Local angular Hessian can be used to improve wave equation least square migration images. By decomposing the original Hessian operator into the local wavenumber domain or the local angle domain, the least square migration image is obtained as the solution of a linearized least-squares inversion in the frequency and local angle domains. The local angular Hessian contains information about the acquisition geometry and the propagation effects based on the given velocity model. The inversion scheme based on the local angular Hessian avoids huge computation on the exact inverse Hessian matrix. To reduce the instability in the inversion, damping factors are introduced into the deconvolution filter in the local wavenumber domain and the local angle domain. The algorithms are tested using the SEG/EAGE salt2D model and the Sigsbee2A model. Results show improved image quality and amplitudes.

Keywords: Hessian, Imaging, Inversion, Least-squares, Local angle domain.

1 INTRODUCTION

The migration process can be formulated as an adjoint operator of the forward modelling operator. Therefore, amplitude distortion in the prestack migration, caused by the limited acquisition geometry, frequency band limitation and complex overburden, can be treated by an inversion problem based on the minimization of a least-squares functional. Beylkin (1985) viewed migration as an inverse Radon transform and derived an amplitude-preserving least-squares migration algorithm. Lailly (1983) and Tarantola (1984) illustrated the equivalence of the migration operator and the gradient operator of the least-squares functional. Symes (2008) presented a liner least-squares migration algorithm and applied it to the reverse time migration process.

To solve the least-squares problem in the linearized case, we need to compute and invert the Hessian, the matrix of the second derivatives of the error functional with respect to the model parameters like velocity or scattering coefficient. However, the full space-domain Hessian of the least-squares functional is so big that we can not afford the cost of the inversion of the exact Hessian matrix for complex media. Therefore, diagonal inverted Hessian is widely used to approximate the exact one (Shin, Jang and Min 2001; Rickett 2003; Plessix and Mulder 2004; Symes 2008). Under the assumption of the high-frequency approximation and infinite acquisition aperture, the Hessian matrix is diagonal. However, for true seismic acquisition, the Hessian is not diagonal and not even diagonally dominant (Pratt, Shin and Hicks 1998; Plessix and Mulder 2004; Tang 2009; Ren, Wu and Wang 2009). In many cases, one scattering point in the model space will spread into an elliptic region in the image space represented by a column vector in the Hessian matrix.

Several techniques have been developed to calculate the inverted Hessian in an approximation sense. Yu et al. (2006) computed the inversion of the Hessian in $v(z)$ media. Lecomte (2008) computed the inverted Hessian using a ray-based approach. However, these approximations for the media or the acquisition systems may not reflect the real case of seismic...
data. Plessix and Mulder (2004) introduced a wave-equation based approach to generate the Hessian matrix in the space domain. They gave four different approximations for the diagonal Hessian. Tang (2009) developed the space-domain Hessian calculation using phase-encoding techniques. Herrmann et al. (2009) developed a curvelet-based migration preconditioning and scaling method. Another way of generating the wave-equation based Hessian operator is to implement it in the phase-space or the local angle domain. Local exponential frame (Mao and Wu 2007) beamlet decomposition can eliminate directional ambiguity in the local cosine basis beamlets (Wu, Wang and Luo 2008). Cao and Wu (2005) and Xie, Jin and Wu (2006) used a local plane-wave decomposition method to generate the phase-space amplitude correction factor using a one-way wave equation approach.

In this paper, we use the local cosine basis beamlet migration and the local exponential frame decomposition to calculate the angular components of the Green’s function. We propose two schemes for the decomposition of Hessian in the local wavenumber domain or in the local angle domain. The first one is to decompose both the exact Hessian and the migration stacked image into the local wavenumber domain. This method is a direct implementation of spectral decomposition of both the image and the Hessian into the local wavenumber domain. However, this method is time-consuming. To develop a fast local angular Hessian computation method, in the second approach we compute the Hessian matrix in the local angle domain using the angular components of the Green’s function based on beamlet migration in the frequency domain. Taking stability into consideration, damping factors are used in the dip-angle domain Hessian operator. This method is easy to expand to the 3D case after the 3D dip-angle dependent imaging result is produced.

This paper is organized as follows. Firstly, we review the linear least-squares formulation and its relation with the Born approximation. Then, we propose and study two Hessian matrix decomposition methods. Finally, we test the methods using the data sets of the SEG/EAGE salt2d model and Sigsbee2A model. When we compare the diagonal Hessian compensation result, our methods give better compensation results.

2 LEAST SQUARE MIGRATION AND THE HESSIAN

Under the Born approximation, the scatterings of each volume element are independent from each other. The scattered field can be written as a superposition of scattered waves from all elements,

\[ d_m(x_i, x_j, \omega) = \omega^2 \sum_{x_k} G(x_i, x_k, \omega) m(x_k) G(x_k, x_j, \omega), \]

where \( d_m(x_i, x_j, \omega) \) is the simulated seismic data in the frequency domain recorded at the receiver point \( x_j \) excited by a shot at point \( x_i \) on the surface. For each imaging point \( x_i \), \( G(x_i, x_j, \omega) \) and \( G(x_k, x_j, \omega) \) are the monochromatic Green’s function from \( x_k \) to \( x_j \) and from \( x_i \) to \( x_k \) in the inhomogeneous background media respectively. \( m(x_j) \) is the slowness perturbation function of the subsurface velocity. Equation (1) can also be written into an operator form,

\[ \mathbf{d}_m = \mathbf{L} m(x_j). \]

where \( \mathbf{L} \) is the linear operator relating the medium perturbation vector \( m(x_j) \) to the data \( \mathbf{d}_m \) and the elements of \( \mathbf{L} \) for each shot, each receiver and each imaging point \( l(x_i, x_j, x_k) = \omega^2 G(x_i, x_k, \omega) G(x_k, x_j, \omega) \). Here we use the vectors to stand for collections of variations. For an imaging or inversion problem, \( m(x_j) \) is the inversion parameter for all the imaging points \( x_j \). A misfit functional is formulated as,

\[ f(m) = ||\mathbf{L} m - \mathbf{d}_{\text{obs}}||. \]

where \( \mathbf{d}_{\text{obs}} \) is the observed data vector, which is different from \( \mathbf{d}_m \) predicted by the given model in equation (2); and \( || \cdot || \) stands for the L2 norm. Under the linear assumption, the optimal slowness perturbation \( m^{opt} \) satisfies \( \mathbf{d}_{m^{opt}} \approx \mathbf{d}_{\text{obs}} \) and,

\[ \mathbf{H} m^{opt} = -\mathbf{g}. \]

where \( \mathbf{g} = -\mathbf{L}^* \mathbf{d} \) is the gradient of \( f \) evaluated at \( m = 0 \) and \( \mathbf{L}^* \) is the adjoint operator of \( \mathbf{L} \). \( \mathbf{H} = \mathbf{L}^* \mathbf{L} \) is the second derivative of the error functional, called the Hessian matrix. Therefore, a one-time linear iteration LS solution is given by,

\[ m^{opt} = -\mathbf{H}^{-1} \mathbf{L}^* \mathbf{d}_{\text{obs}}. \]

\( \mathbf{L}^* \mathbf{d}_{\text{obs}} \) is equivalent to the stack of the migrated shot sections (Lailly 1983). If an exact inverse Hessian can be found, equation (5) will lead to a least-squares solution \( m_1 \). The gradient term and the Hessian term can be written as (Plessix and Mulder 2004),

\[ g(x_j) = \Re\left( \sum_{\omega} \omega^2 \sum_{x_k} f_i(\omega) G(x_i, x_j, \omega) G(x_k, x_j, \omega) \right. \]

\[ \times \left( d_m^*(x_i, x_j, \omega) - d_{\text{obs}}^*(x_i, x_j, \omega) \right), \]

© 2011 European Association of Geoscientists & Engineers, Geophysical Prospecting, 59, 651–661
Figure 1 Compensation result using the diagonal Hessian matrix. a) Migration stacked image using the LCB operator. b) Diagonal Hessian of the SEG/EAGE salt2D model. c) Compensation result using the diagonal Hessian.

\[ H(x_i, x_j) = \text{Re} \left( \sum_\omega \omega^4 \sum_{x_s} |f_s(\omega)|^2 |G(x_s, x_i, \omega)|^2 G^*(x_s, x_j, \omega) \right) \times \sum_{x_r} |G(x_i, x_r, \omega)|^2 |G(x_r, x_j, \omega)|^2. \]  

(7)

Figure 1 shows the compensation result on the SEG/EAGE salt2d model using the diagonal Hessian. From the figure of the shot migration stacked image based on the one-way wave-equation local cosine basis propagator (see Fig. 1a) and the compensation result by the inversion of the diagonal-Hessian matrix (see Fig. 1b), we see that the image amplitude in the subsalt weakly illuminated area is enhanced compared to the original image, although the improvement is limited.

However, in many cases, the off-diagonal elements of the Hessian matrix are proved to be useful in the multiples and the non-full covered acquisition system. The system response for an imaging point is a volume distribution near the scattering point and is a column vector in the exact Hessian matrix. The Hessian responses (the column vectors) for points in the

© 2011 European Association of Geoscientists & Engineers, Geophysical Prospecting, 59, 651–661
SEG/EAGE salt2D model are shown in Fig. 3, corresponding to the selected point in the velocity model shown in Fig. 2. We find that the Hessian vectors spread into ellipse areas and have directional character. The directional character corresponds to the local directional illumination pattern, which gives the physical basis for the compensation in the local angle domain.

3 ANGULAR HESSIAN AND IMAGING IN THE LOCAL ANGLE DOMAIN

We can calculate the Hessian following equation (7). The Green’s function can be generated by any forward modelling method. By analysis above, only those surrounding points \( x_j \) near the imaging point \( x_i \) contribute significantly to \( H(x_i, x_j) \). Therefore, we just need to calculate the Hessian for each point in its local area. To see the directional character, firstly we generate the exact Hessian and apply the local wavenumber domain decomposition to the imaging field and the Hessian. This implementation is time-consuming but is intuitive for the space-domain Hessian. To develop a fast local angular Hessian computation method, in the second method, we compute the Hessian matrix in the local angle domain directly using the angular components of the Green’s function based on the beamlet migration in the frequency domain.

3.1 The local wavenumber domain Hessian

Equation (4) has been proved to be a convolution under the local homogeneous assumption (Schuster and Hu 2000; Wu et al. 2006), which is also the base of our local decomposition methods. Here, we apply the local wavenumber domain Hessian decomposition for the purpose of image amplitude compensation. For a space-localized area, we transform both the exact Hessian and the imaging field into the local wavenumber domain,

\[
\hat{H}(x_i, k^l) = \sum_{x_j} w(x_j - x_i) H(x_i, x_j) \exp(ik^l(x_j - x_i)).
\]  

In equations (9) and (10), \( w(x_j - x_i) \) is a window function to localize a spatial field, which is applied to the imaging point \( x_i \). The vector \( k^l \) stands for the local wavenumber vector corresponding to the given space window. Here, we compensate the imaging for each wavenumber separately. In view of stability, the damping factor is introduced. The final stacked image after imaging correction in the local wavenumber domain is obtained by,

\[
I(x_i, k^l) = \sum_{k^l} \frac{\hat{H}(x_i, k^l)}{H(x_i, k^l) + \varepsilon(x_i)},
\]

where \( \varepsilon(x_i) \) is a damping factor for the given imaging point to avoid the singular value of Hessian in the wavenumber domain. This de-convolution process in the local wavenumber domain is also similar to the procedure of diffraction tomography (Wu 2007). The de-convolution in equation (5) becomes the division in the wavenumber domain. Thus the compensation result \( I(x_i) \) is equal to the least-squares solution \( m_1 \) in equation (5). This scheme is for the Hessian after the multifrequency stacking process and is a stable inversion. We will show the wavenumber domain Hessian and the compensation result in the numerical test part.

3.2 The local angle domain Hessian using beamlet decomposition

The above mentioned method to generate a wavenumber domain Hessian is time-consuming because it is implemented after the calculation of the exact space-domain Hessian. So we would like to find a fast local angular Hessian calculation method. Local angle domain decomposition can transform the wavefield from the space domain into the local angle domain, which is a spatially varying angle decomposition. Using the local exponential frame (Mao and Wu 2007), we can transform the Green’s function in the space domain to the local

© 2011 European Association of Geoscientists & Engineers, Geophysical Prospecting, 59, 651–661
Figure 3 Hessian responses of selected points. Panels a-f show point Hessian responses corresponding to the locations marked in Fig. 2. Window size is 0.73 km $\times$ 0.73 km around the target points.

angle domain (see Appendix A). The Green’s function can be decomposed as the summation of the discrete angular components,

$$G(\theta_i, x_s; x_r, \omega) = \text{LEF}(G(x_i; x_s, \omega)),$$

where $G(x_i; x_s, \omega)$ is the space-domain Green’s function and $G(\theta_i, x_s; x_r, \omega)$ is its local angle-domain decomposition at $x_i$. Using the angular decomposition of the back-propagation Green’s function in equation (7), we obtain,

$$H(x_i; \theta_i) = \text{Re}\left(\sum_{\omega} \omega^3 \sum_{x_s} |f_s(\omega)|^2 G(x_i; x_s, \omega)G^*(x_i, \theta_i; x_s, \omega)\right)$$

$$\times \sum_{x_r} G(x_i; x_r, \omega)G^*(x_i, \theta_r; x_r, \omega),$$

where $G(x_i; \theta_i; x_s, \omega)$ and $G(x_i, \theta_i; x_r, \omega)$ are the angular components of the Green’s functions for the source and receiver respectively, decomposed centred at $x_i$. $\theta_i = (\theta_s + \theta_r)/2$ is the migration dip-angle. The dependence of $G_s$ and $G_r$ on $x_i$ becomes the dependence to $\theta_s$ and $\theta_r$ after the local exponential frame decomposition. The corresponding image is also decomposed into the dip-angle dependent $I(x_i, \theta_i)$ (Wu et al. 2004). Then the application of the Hessian operator becomes (the same deconvolution as the local wavenumber domain method),

$$I_c(x_i) = \sum_{\theta_i} \frac{I(x_i, \theta_i)}{H(x_i, \theta_i) + \epsilon(x_i)}.$$

So, the direct local angular Hessian calculation scheme can be implemented as follows:

1. For each selected frequency, calculate the Green’s function and the angular components of the Green’s function for each shot and receiver points $G(x_i, \theta_s; x_r, \omega)$;

Figure 4 Hessian, image and the corresponding spectra at a selected point $[20.36, 3.243]$ (km). a) The point Hessian absolute value; b) the local spectrum of point Hessian; c) the real part of the initial local migration image; d) the spectrum of the local image; e) the image after compensation.
For a given $\theta_i$, loop over $\theta_s$ and $\theta_r$ using the relation $\theta_i = (\theta_s + \theta_r)/2$; 3) For each pair of shot point $x_s$ and receiver point $x_r$, choose the corresponding Green’s functions in the local angle domain and calculate the local angular Hessian according to equation (14). The image compensation is done using equation (15).

4 NUMERICAL EXAMPLES

The SEG/EAGE salt2D model is chosen to test the amplitude compensation scheme using the local angular Hessian. Because the subsalt areas are weakly and unevenly illuminated, standard one-way migration can not give balanced image amplitudes, as shown in Fig. 1.
4.1 Amplitude compensation using the local wavenumber domain Hessian

For the amplitude compensation, firstly, we calculate the Green’s functions using a one-way local cosine basis operator (Wu et al. 2008), other operators can also be used to calculate the Green’s function only if they are used for the migration process. To test our methods, all the points’ Green’s functions with dense frequency sampling are calculated.

Based on equation (6), we first calculate the frequency-dependent Hessian in the space domain. A 30 × 30 grid space window is used to localize the Hessian. Other elements outside the window that are of no use for the angular Hessian calculation are neglected. 186 frequencies are sampled from 1–35 Hz to calculate the space-domain exact Hessian. Here the reciprocity theorem is used to calculate the propagation from the reflector to the receiver:

\[ G(x_i, x_r, \omega) = G(x_r, x_i, \omega). \]

To test the local wavenumber domain Hessian, we calculate the local Hessian and imaging result in the space domain and transfer into the local wavenumber domain. One imaging point is shown in Fig. 4. The compensation result for the whole model is shown in Fig. 5. The damping factor varying with depth (from two per cent at the first layer to 0.5 per cent at the deepest layer) is used. From the figure, we can see the structure of the image is more continuous and the amplitude is more balanced after the compensation, especially for the faults. However, this full implementation from the space domain into the local wavenumber domain for the exact Hessian of all the frequencies is very time-consuming. For the SEG/EAGE salt2D model, we need 65 hrs to generate the exact Hessian and another 10 hrs to calculate the local wavenumber domain Hessian on paralleled 20 nodes. This implementation must calculate a dense sampled wavenumber because the image part will lose information if we use sparse wavenumber sampling.

4.2 Amplitude compensation using the local angle domain Hessian

For the second method, the image in the local angle domain can be obtained naturally by the beamlet migration. After the one-way local cosine basis migration, images can be easily sorted into dip-angle dependent gathers by the local exponential frame. Besides, the local angular Hessian is frequency dependent. We just need to calculate the Hessian for a few frequencies. This will lead to a fast local angular Hessian calculation implementation. We can separate the image field into a series of frequency bands and compensate using the central frequency local angular Hessian for each band. Here we calculate the local angular Hessians for 6 frequencies separately (5 Hz, 10 Hz, 15 Hz, 20 Hz, 25 Hz, 30 Hz). In Fig. 6, the angular Hessians of the main frequency 15 Hz for the points in Fig. 3 are shown. There is correspondence between the local angular Hessian and their space-domain Hessian, as shown in Fig. 3. Angular Hessian estimates the point spreading as a function of angle, which represents the directional effects of the acquisition system. Some noises are introduced into the image as shown in Fig. 7. However, the final compensation result has higher resolution and the faults beneath the salt are better imaged. At the same time, the multiples under the salt shown in the original image (Fig. 1) are suppressed significantly. Shown in Fig. 8 are the imaging amplitudes of the bottom horizontal line in the SEG/EAGE salt2d model. From the figure, we can see that after compensation the amplitudes are more balanced than the original image. The best compensation result is the one compensated by the local angular Hessian because both the Hessian and the image are dip-angle dependent. At the same time, using the local angular Hessian method, the computation time is much less than the first method. In this test, we just need 1.5 hrs to generate the local angular Hessian using 20 nodes. Another benefit is we do not need to store huge data for the large exact Hessian.
Due to the efficiency of the angular Hessian method, we can apply the method to large models, such as the synthetic Sigsbee2A data set, which is based on the geologic setting found in the deepwater Gulf of Mexico. The velocity model has a trace sampling interval of 11.43 m and a depth sampling interval of 7.62 m. Figure 9 gives the compensation result using the local angular Hessian with the main frequency. From the compensation, the subsalt scattering points are imaged well and the amplitudes are much more balanced than before.

5 DISCUSSIONS AND CONCLUSIONS

We present two methods to calculate the Hessian matrix of least-squares migration in the local wavenumber domain or the local angle domains. The first method directly decomposes the imaging field and the exact Hessian into the local wavenumber domain. The wavenumber domain de-convolution is proved to be effective using the SEG/EAGE salt2D model. Images after the compensation with the local wavenumber domain Hessian have a much more balanced image. Taking consideration of efficiency, in the second method we decompose the back-propagated Green’s function to the local angle domain and then calculate the angular components of the exact Hessian. The local angular Hessian can be used as migration weights in the local angle domain to improve the image quality and amplitude in regions with complex overburden, such as in the subsalt areas. Through numerical examples on the SEG 2D salt model and Sigsbee-2A model, we have demonstrated that the methods of local angular Hessian are superior than the diagonal Hessian and are effective in obtaining more balanced and continuous subsalt images. The local angular Hessian is similar to the acquisition amplitude factor proposed by Wu et al. (2004). We give a comparison of the local angular Hessian and the acquisition amplitude factor in Appendix B.

ACKNOWLEDGEMENTS

The authors would like to thank the reviewers for their beneficial comments to our paper. H. Ren and R.-S. Wu would like to thank the sponsors of WTOPI (Wavelet Transform On Propagation and Imaging for seismic exploration), Research Consortium at University of California, Santa Cruz for their financial support. H. Ren and H. Wang would like to thank for financial support by the China Important National Science & Technology Specific Projects (2008ZX05023-005, 2008ZX05005-005) and support by the Sinopec Geophysical
Figure 9 Angular Hessian compensation on the Sigsbee2A model. a) Shot migration stacked image result using a local cosine basis propagator; b) compensation result using the local angle domain Hessian.

References


© 2011 European Association of Geoscientists & Engineers, Geophysical Prospecting, 59, 651–661
APPENDIX A: LOCAL EXPONENTIAL FRAME TO DECOMPOSE THE WAVEFIELD INTO LOCAL ANGLE DOMAIN

To decompose the wavefield into the local wavenumber domain or the local angle domain, we can use the local exponential frame, which is combined by the local cosine/sine basis elements (Mao and Wu 2007):

\[ b^{(c)}_{mn}(x) = \sqrt{\frac{2}{L_m}} B_n(x) \cos \left( \pi \left( m + \frac{1}{2} \right) \frac{x - \bar{x}_n}{L_m} \right), \]

\[ 0 \leq m \leq M - 1, \] (A1)

\[ b^{(s)}_{mn}(x) = \sqrt{\frac{2}{L_m}} B_n(x) \sin \left( \pi \left( m + \frac{1}{2} \right) \frac{x - \bar{x}_n}{L_m} \right), \]

\[ 0 \leq m \leq M - 1, \] (A2)

where \( \bar{x}_n \) is the characterization position for the local decomposition. The space window length \( L_m = \bar{x}_{n+1} - \bar{x}_n \), where \( n \) stands for the serial number of the window. \( B_n(x) \) is a bell function in the compact interval \( [\bar{x}_n - \varepsilon, \bar{x}_{n+1} + \varepsilon] \), with \( \varepsilon \) as the overlapping radius. \( m \) stands for the serial number of the local wavenumber \( k_m = \frac{\pi (m + 1/2)}{L_m} \). We combine equations (A1) and (A2) into,

\[ g_{mn}(x) = \sqrt{\frac{2}{L_m}} B_n(x) \exp \left( ik_m(x - \bar{x}_n) \right), \]

\[ -M + 1 \leq m \leq M - 1. \] (A3)

We call \( g_{mn}(x) \) the local exponential beamlet that forms a tight-frame of redundancy 2. The frequency domain wavefield in one propagation layer \( u(x, \omega) \) can be decomposed into local exponential beamlets with windows along the x-axis,

\[ u(x, \omega) = \sum_n \sum_m \hat{u}_{mn}(\bar{x}_n, k_m, \omega) g_{mn}(x) \]

\[ = \sum_m \exp \left( ik_m x \right) \sum_n \hat{u}_{mn}(\bar{x}_n, k_m, \omega) \sqrt{\frac{2}{L_m}} B_n(x) \exp \left( -ik_m \bar{x}_n \right) \] (A4)

Then we have a partial reconstructed wavefield in the local wavenumber domain,

\[ u(x, k_m, \omega) = \exp \left( ik_m x \right) \]

\[ \times \sum_n \hat{u}_{mn}(\bar{x}_n, k_m, \omega) \sqrt{\frac{2}{L_m}} B_n(x) \exp \left( -ik_m \bar{x}_n \right). \] (A5)

The full reconstructed wavefield can be summed up by all the local wavenumber domain wavefields by equation (A5). For the local wavenumber \( k_m \), the corresponding propagating angle is,

\[ \theta_m = \arcsin \left( \frac{v(x)}{\omega} k_m \right). \] (A6)

Here, \( v(x, z) \) is the velocity of the wavefield point and \( \theta_m \) is the \( m^{th} \) sampled angle. Using equations (A5) and (A6), we can construct the wavefield both in the local wavenumber domain and in the local angle domain and the local angle domain wavefield has the character of equation (13).

APPENDIX B: COMPARISON OF THE LOCAL ANGULAR HESSIAN AND ACQUISITION AMPLITUDE FACTOR

Here we would like to discuss the relation between the Hessian matrix based approach for image amplitude correction and the acquisition aperture correction of Wu et al. (2004). The local angular Hessian derived in this paper is closely related to the
acquisition-aperture amplitude factor (Wu et al. 2004),
\[ |F_a(\bar{x}, \bar{\theta}_s, \bar{\theta}_g)| = \sum_{x_i} |G^*_i(\bar{x}, \bar{\theta}_s; x_i)G_{Ap}(\bar{x}, \bar{\theta}_s; x_i)||B_A(\bar{x}, \bar{\theta}_g)|, \]  
\[ \tag{B1} \]
\[ |B_A(\bar{x}, \bar{\theta}_g)|^2 = \left| \int_{\partial(e)} dx_g G_F(\bar{x}, \bar{\theta}_g; x_g) \frac{\partial G^*_i(\bar{x}, \bar{\theta}_g; x_g)}{\partial z} \right|^2 \]
\[ = \int_{\partial(e)} dx_g |G_F(\bar{x}, \bar{\theta}_g; x_g)|^2, \]  
\[ \tag{B2} \]
where \( G_F \) is the Green’s function for forward modelling and \( G_i \) is the Green’s function for the imaging (back-propagation). Equation (B2) represents the scattered energy received by the receiver array for a given source, which is based on energy conservation and assuming that the back-propagation process (the focusing operator) for migration can focus all the received energy back to the scattering point. We can see the close similarity of the two formulations if we use the same Green’s function for both modelling and imaging. However, these are based on different principles and have the following differences:
1. For the Hessian based on the Born model, there is a \( k^2 \) factor in the modelling operator, as for the adjoint operator, resulting in a \( \omega^4 \) factor in the Hessian; while for acquisition-aperture correction the focusing operator does not have the \( \omega^2 \) factor, resulting in different frequency dependences of the two formulations.
2. In the gradient operator or the Hessian, the adjoint operator has the same Green’s function as in the modelling operator but phase-conjugated; while in the focusing operator (migration operator) the normal gradient of the Green’s function is used, resulting in a different radiation pattern and different geometric spreading.
3. The acquisition-aperture correction of Wu et al. (2004) considers only the amplitude effect and is a one-step operation; while Hessian is based on the least-square inversion and can be used for iterative inversion.