THEORY OF TRANSMISSION FLUCTUATIONS IN RANDOM MEDIA WITH A DEPTH-DEPENDENT BACKGROUND VELOCITY STRUCTURE

YINGCAI ZHENG AND RU-SHAN WU

ABSTRACT

An extended theory on the coherence function of log amplitude and phase for waves passing through random media is developed for a depth-dependent background medium using the WKBJ-approximated Green’s function, the Rytov approximation, and the stochastic theory of the random velocity field. The new theory overcomes the limitation of the existing theory that can only deal with constant background media. Our extended coherence functions depend jointly on the angle separation between two incident plane waves and the spatial lag between receivers. The theory is verified through numerical simulations using the iasp91 background velocity model with two layers of random media. The current theory has the potential to be used to invert for the depth-dependent spectrum of heterogeneities in the Earth.

KEY WORDS: Transmission fluctuation, coherence function, random media, heterogeneity spectrum, Phase, amplitude, WKBJ, Rytov © 2008 Elsevier Inc.

1. Introduction

The statistical approach, complementary to the deterministic method such as seismic tomography, is indispensable in probing the small-scale inhomogeneities in the Earth using scattered seismic waves. Its characterization of heterogeneities using statistical parameters has yielded many important physical constraints and interpretations for the lithosphere to the core-mantle boundary region, and led to key implications regarding the Earth’s compositional constituents, thermal state, dynamic mixing, and others.

In 1970s, many attempts (Capon, 1974; Capon and Berteussen, 1974; Berteussen, 1975; Berteussen et al., 1975, 1977) had been made to infer the spatial spectrum of the velocity heterogeneity using observed fluctuations of the logarithmic amplitude \( \log A \) and the phase \( \phi \) across a seismic array since the pioneering work by Aki (1973). Aki essentially employed the Chernov theory (Chernov, 1960) to study the transverse coherence function (TCF) using the data from the Large Aperture Seismic Array (LASA), Montana. He found that a 60-km random layer in the lithosphere with \( \sim 10 \) km scale length for the heterogeneities and \( 4\% \) root-mean-square (rms) \( P \)-wave velocity fluctuation could explain the data. The Chernov theory (1960) studies wave propagation in a single layer of stationary random velocity heterogeneities with a Gaussian correlation function and with a constant background velocity.
and as such, the TCF has a nice closed mathematical form. By the word stationary, we mean the spatially translational invariance of the statistic (e.g., correlation function). The TCF for random media of general spectral type can be found in Tatarskii (1971) and Ishimaru (1978) and it does not possess a simple analytical expression in general. It is interesting to note that this stochastic approach predated the deterministic tomography method (Aki and Lee, 1976; Aki et al., 1976, 1977).

The heterogeneity spectrum of the Earth is fractal based on well-logging data (Wu et al., 1994b; Jones et al., 1997; Goff and Holliger, 1999) and it may not be Gaussian. Capon and Bertussessen (1974) found that the Chernov theory was not applicable to fluctuations of logA and phase data under the Norwegian Seismic Array (NORSAR); however, they attributed this to the validity of the Born approximation at high frequencies involved in the theory. The Earth’s large-scale structures are largely stratified in depth and the herteogeneity spectrum can be slowly varying with depth. Flatté and Wu (1988) introduced a new kind of coherence function, called the angular coherence function (ACF), to resolve the depth-dependent spectra under the NORSAR. A two-layer model with power-law type medium was favored over a one-layer stationary Gaussian medium. Wu and Flatté (1990) also formulated the joint transverse and angular coherence function (JTACF) in which both the spatial lag between two seismic stations and the angular separation between two plane waves were taken into account. Their formulation was based on wave perturbation theory and assumed the Rytov and the parabolic approximations for the wave equation. Chen and Aki (1991) independently obtained the same JTACF result using the Born approximation. Parametric (Flatté et al., 1991a) and nonparametric (Wu and Xie, 1991) inversions have been carried out to process real data or to study the depth resolution in the spectral inversion. Wu and Xie (1991) called such inversion “stochastic tomography” and they found that the JTACF has the best depth resolution, that the ACF has limited depth resolution close to the surface, and that the TCF has no depth resolution at all. To invert for the spectrum of a single-layer stationary random heterogeneity, Zheng et al. (2007) proposed a new scheme using only the TCF data for logA and phase, in which a Fourier transform was established between the heterogeneity spectrum and the combination of the logA and phase TCF data. There was some concern on the discrepancy on the phase coherence function between the theory and the numerical simulation (Line et al., 1998a; Hong et al., 2005). However, recent investigation has shown that this discrepancy was caused by the incorrect phase picking method used in the numerical and field experiments. Application of the correct phase measuring method has resulted in excellent agreement between numerical tests and the theory (Zheng and Wu, 2005). The coherence function formation using combined data from arrays with different apertures was investigated by Flatté et al. (1991b) and Flatté and Xie (1992). The theory of transmission fluctuation was also applied to seismic reflection data to obtain heterogeneities in the upper crust (Line et al., 1998b). The theory of JTACF has been applied to the NORSAR data (Wu et al., 1994a) and the Southern California Seismic Network data (Liu et al., 1994; Wu et al., 1995). For earlier reviews on the subject, see Wu (2002) and Sato et al. (2002).

Besides the coherence function study, many other seismological methods have been devised to characterize the small-scale heterogeneities. The seismic coda envelope analysis (Sato and Fehler, 1998) has been widely used to investigate the scattering strength in the Earth. Through studying the radiated power carried by precursors to the PKIKP phase (Cleary and Haddon, 1972), the spectrum for P-wave volumetric scatters in the D00 region, or for the core-mantle boundary topographical relief (Bataille and Flatté, 1988; Bataille et al., 1990), or for the whole mantle (Hedlin et al., 1997) and mid-mantle (Hedlin and Shearer, 2002) has been constrained.
Despite significant progress that has been made using the coherence function to characterize the Earth’s small-scale heterogeneities, all these theoretical treatment of the problem is based on wave propagation in a homogeneous background medium. This is inadequate for the real Earth, which has a depth-dependent velocity profile to first order. In this chapter, we show how we can generalize the theory of coherence functions (log $A$ and phase $\phi$) to arbitrary random medium superimposed on general depth-dependent background medium. This generalized theory can be directly utilized for the real Earth.

2. Acoustic Waves in Stratified Media and WKBJ Green Function

The linearized wave equation for pressure in the frequency domain is

$$\rho \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) + \frac{\omega^2}{c^2} p = 0.$$  (1)

Let $\omega$ be the angular frequency, $p = p(x,y,z,\omega)$ the pressure field, and $\nabla$ the spatial gradient operator. Define $z$ as depth variable, $\rho$ and $c$ are density and wave propagation speed, respectively. For a stratified medium, $c = c(z), \rho = \rho(z)$, the 2-D Fourier transform can be performed to Eq. (1) with respect to spatial coordinates, $x$ and $y$, to obtain

$$k_z^2(z)p + \frac{\partial^2 p}{\partial z^2} - \rho^{-1} \frac{\partial \rho}{\partial z} \frac{\partial p}{\partial z} = 0,$$  (2)

where $k_z$ is the vertical wave number defined as

$$k_z(z) = \pm \sqrt{\frac{\omega^2}{c^2}(z) - k_x^2 - k_y^2}.$$  (3)

$k_x$ and $k_y$ are horizontal wave numbers corresponding to $x$ and $y$, respectively. The plus sign in Eq. (3) corresponds to the downgoing wave and the minus sign the upgoing wave. We use the same symbol $p$ to denote the pressure before and after the Fourier transform and this should not cause confusion. Plugging the trial solution in the form of $p = A(z)\exp[i\phi(z)]$ into Eq. (2) and the real part is

$$k_z^2A + \left( A'' - \phi'^2A \right) - \rho^{-1} \frac{\partial \rho}{\partial z} A' = 0,$$  (4)

and the imaginary part reads

$$2\phi'A' + A\phi'' - \rho^{-1} \frac{\partial \rho}{\partial z} A\phi' = 0.$$  (5)

The prime represents the partial derivative with respect to depth $z$. If the amplitude is slowly varying with depth, we can use the WKBJ approximation (i.e., get rid of all terms that involve derivatives of $A$) in Eq. (4) and it reduces to

$$k_z^2 - \phi'^2 = 0 \Rightarrow \phi' = k_z.$$  (6)
Therefore, the phase function is solved as

$$\phi(z) = \int_{z_s}^{z} k_z(z')dz', \quad (7)$$

where $z_s$ is a reference depth, commonly taken as the source depth. The validity of the WKBJ approximation is assured if the wavelength is shorter than the characteristic scale of the background velocity model. The WBKJ also implies energy conservation for the transmitted wave, which means that no reflected waves are produced (Wu and Cao, 2005). Substituting Eq. (6) in Eq. (5), we have

$$2 \frac{d \ln A}{dz} + \frac{d \ln k_z}{dz} - \frac{d \ln \rho}{dz} = 0, \quad (8)$$

Rearranging Eq. (8), we get

$$d \ln A = d \ln \sqrt{\frac{\rho(z)}{k_z(z)}}, \quad (9)$$

So the solution for the amplitude in Eq. (9) is

$$A(z) = C \sqrt{\frac{\rho(z)}{k_z(z)}}, \quad (10)$$

where $C$ is a constant. The final solution to Eq. (2) is

$$p(k_x, k_y, z, \omega) = C \sqrt{\frac{\rho(z)}{k_z(z)}} \exp\left[i \int_{z_s}^{z} k_z(z')dz'\right]. \quad (11)$$

Once we have obtained Eq. (11), the solution for Eq. (1) is just the inverse Fourier transform

$$p(x, y, z, \omega) = \left(2\pi\right)^{-2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(k_x, k_y, z, \omega) \exp[i(k_x x + k_y y)] dk_x dk_y. \quad (12)$$

Next, let us investigate the solution for a point source. Under this case, the Eq. (1) has an additional source term

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla p\right) + \frac{\omega^2}{c^2} p = -\delta(x - x_s)\delta(y - y_s)\delta(z - z_s). \quad (13)$$

$x_s, y_s$ and $z_s$ are source position coordinates. For simplicity, the source is placed at the origin. In the homogeneous case, the solution in frequency domain is

$$p(x, y, z, \omega) = \frac{1}{4\pi r} \exp\left(\frac{i\omega r}{c}\right), \quad r = \sqrt{x^2 + y^2 + z^2}. \quad (14)$$
When a receiver is sufficiently close to the source, the solution (12) should coincide with (14). In order to compare both solutions in the wave number domain, we first expand the point source solution (14) into plane wave components using the Weyl integral (e.g., Aki and Richards, 2002 p. 190). The coefficient to component \( k_x \) is \( 2 \frac{i \sqrt{\rho(z_s)/k_z(z_s)}}{2} \) and the corresponding coefficient in Eq. (12) is \( C = \sqrt{\rho(z_s)/k_z(z_s)} \). Clearly, these two should be equal. Thus, this constant \( C \) is

\[
C = \frac{1}{2i \sqrt{\rho(z_s)k_z(z_s)}}.
\] (15)

Therefore, the complete Green’s function for both 3-D and 2-D cases can be obtained. Let us rewrite the Green’s function in 3-D case

\[
G(x, y, z, \omega) = \frac{(2\pi)^{-2}}{2i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2i \sqrt{k_z(z_s)k_z(z)}} \sqrt{\frac{\rho(z)}{\rho(z_s)}} \exp \left\{ i \left[ k_x x + k_y y + \int_{z_s}^{z} k_z(z') dz' \right] \right\} dk_x dk_y.
\] (16)

In 2-D case, just drop all terms pertaining to \( k_y \) and the inverse Fourier transform constant is \( (2\pi)^{-1} \). Also remember that the 2-D Fourier transform becomes a 1-D transform.

3. **Rytov Solution to the Wave Equation in a Heterogeneous Medium**

For convenience, we rewrite the monochromatic wave equation

\[
\rho \nabla \cdot (\rho^{-1} \nabla p) + \frac{\omega^2}{c^2} p = 0.
\] (17)

The solution for the pressure wavefield is sought in the form of

\[ p = p_0 e^{\psi}. \] (18)

\( \psi = \psi(x', \omega) \) is the complex phase and \( p_0 = p_0(\omega, x') \) is the background incident wavefield. In what follows we suppress the explicit dependence on \( \omega \) for the wavefield for simplicity in notation. Assuming no lateral density perturbation and substituting (18) into (17), we have

\[
(\rho \nabla \cdot (\rho^{-1} \nabla p)) (p_0 \psi) = -2k^2 \gamma p_0 - p_0 \nabla \psi \cdot \nabla \psi,
\] (19)

where \( \gamma = (c_0^2/c^2 - 1)/2 \approx -\delta c/c_0 \) is the scattering potential, \( \delta c = c - c_0 \). The background wave number \( k \) is defined as \( k = \omega/c_0(z') \). The Rytov approximation assumes \( (\nabla \psi)^2 \ll 2k^2 \gamma \). The right-hand side of Eq. (19) is a spatially distributed source term. The complex phase can be solved as

\[
\psi(x') = \frac{2}{p_0(x)} \left[ \int k^2(x') G(x', x''', \omega) \gamma(x'') p_0(x'') d^3x'' \right].
\] (20)

From Eq. (20), it can be seen that contributions of scattering potentials \( \gamma \) at different locations to the complex phase are independent from each other. However, this linear
relationship is to the complex phase, not to the field itself, which is different from the Born approximation. Note that this complex phase is naturally “unwrapped.” It can be directly used to obtain the phase velocity. We see that the Rytov approximation is a single-scattering approximation in terms of complex phase. A local Rytov approximation has been developed (Huang et al., 1999) to calculate wavefields, in which case the multiple forward scattering is included.

4. Complex Phase $\psi$ Due to a Plane Wave Incidence

To obtain the complex phase at depth $z$, we can rewrite Eq. (20) in an explicit form

$$
\psi(\bar{x}) = \frac{2}{p_0(\bar{x})} \int \frac{k^2(z')}{z} dz' \int G(\bar{x}_q - \bar{x}_z, z, z', \omega) \gamma(\bar{x}_q', \bar{x}_z') \times p_0(\bar{x}') d\bar{x}_z',
$$

(21)

where $\bar{x} = (\bar{x}_q, \bar{x}_z)$ is the receiver location and $\bar{x}' = (\bar{x}_q', \bar{x}_z')$ is the location for the heterogeneity. In the case of an upgoing plane wave (negative vertical wave number), incidence with ray parameter $\bar{q}$,

$$
\frac{p_0(\bar{x}')}{p_0(\bar{x})} = \sqrt{\frac{\rho(\bar{z})}{\rho(z)}} \sqrt{|k_z(q, z)| \exp \left[ i\omega \bar{q} \cdot (\bar{x}_q' - \bar{x}_q) - i \int_z ^ {\bar{z}} k_z(q, s) ds \right] ,
$$

(22)

where $q = |\bar{q}|$ and $k_z(q, z) = \omega \sqrt{1/c^2(q) - q^2}$. We always choose a nonnegative $\omega \geq 0$ because of the complex conjugate symmetry $p(\omega, \bar{x}) = p^*(\omega, \bar{x})$. Substituting Eqs. (16) and (22) into Eq. (21) and recognizing that it represents a convolution, we can easily put down the complex phase in the wave number domain:

$$
\psi(\bar{q}, \bar{x}) = 2(2\pi)^{-2} \int z d\bar{z} \int k^2(z') \sqrt{k_z(q, z)} \sqrt{k_z(q, z')} \exp \left[ i \int_z ^ {\bar{z}} [k_z(\kappa + \omega \bar{q}, z') - k_z(q, z')] dz' + i \kappa \cdot \bar{x} d\nu(\kappa, \bar{z}) \right] ,
$$

(23)

where $\kappa$ is the horizontal wave number of the heterogeneity spectrum and $d\nu(\kappa, \bar{z})$ is the Fourier-Stieltjes spectral density (Yaglom, 1962). Such a spectral representation is very technical and it seems harmless to replace $d\nu(\kappa, \bar{z})$ by $v(\kappa, \bar{z}) d^2\kappa$, where $v(\kappa, \bar{z})$ is the Fourier spectrum of the velocity perturbation field $\gamma(\bar{x}_q, \bar{x}_z')$ in the transverse plane at depth $z'$. If one wants to pursue the mathematical exactness of the Green function (16), all wave numbers should be integrated, including both propagating (real $k_z$) and evanescent components (imaginary $k_z$). However, the evanescent components are not used in our case. For high-frequency wave propagation in random media, most scattered energy is in forward direction within an angle that spans $-1/(ka)$, with $a$ being the scale length of the heterogeneity. The forward direction is understood as the incoming direction of the
incident wave. If we consider high-frequency wave propagation, Eq. (23) can be simplified as

$$\psi(\vec{q}, \vec{x}) \approx (2\pi)^{-2} \int_0^L \int_{k_c(q, z')} \frac{k^2(z')}{ik_c(q, z')} e^{i\vec{k} \cdot \vec{x} + i\int_0^z [k_c(\vec{R} + \omega \vec{q}, z) - k_c(q, z)] dz} \, dv(\vec{k}, z'),$$  

(24)

where $L$ is the depth of the lower boundary of the heterogeneous layer and receivers are placed at zero depth (Fig. 1). Here we define

$$D(\vec{k}, z') = \frac{1}{\omega} \int_0^{z'} \left[ k_c(\vec{k} + \omega \vec{q}, z) - k_c(q, z) \right] dz.$$  

(25)

Under the forward scattering approximation ($ka > 1$, $k$ is the wave number and $a$ is the characteristic scale length of the heterogeneity), only wave numbers around $\vec{q}$ are integrated. We can expand the $D$ function [Eq. (25)] around $\vec{k} = 0$ using the Taylor expansion,

$$D(\vec{k}, z') \approx -\vec{k} \cdot \vec{r}(\vec{q}, z') + \frac{1}{\omega} \left[ \vec{k} \cdot \vec{q}, \omega, z' \right] + \cdots$$  

(26)

$\vec{r}(\vec{q}, z')$ is the transverse vector between the piercing points at depths 0 and $z'$ for a ray incidence from below with slowness vector $\vec{q}$. The quadratic term of $\kappa$ is

$$\theta(\vec{k}, \vec{q}, \omega, z') = -\frac{1}{2} \frac{I_1(q, z') \kappa^2}{\omega^2} - \frac{1}{2} \frac{I_2(q, z') (\vec{k} \cdot \vec{q})^2}{\omega^2},$$  

(27)

where

$$I_1(q, z') = \int_0^{z'} \left[ c^{-2}(z) - q^2 \right]^{-1/2} dz,$$

(28)

and

![Schematic geometry](image)

**Fig. 1.** Schematic geometry used in the theoretical derivation. The heterogeneous layer is bounded between depth 0 and $L$. Seismic receivers (triangles) are placed on depth zero. Ray 1 connects to station $x_1$, and it has the same slowness as plane wave 1. $\vec{r}_1(\xi)$ is the transverse distance between ray 1 at depth $\xi$ to station $\vec{x}_1$. Similar meaning applies to ray 2 and $\vec{r}_2(\xi)$. 

THEORY OF TRANSMISSION FLUCTUATIONS IN RANDOM MEDIA
\[
I_2(q, z') = \int_0^{z'} \left[ c^{-2}(z) - q^2 \right]^{-3/2} dz. \tag{29}
\]

Obviously, we have \( I_1, I_2 > 0 \). The following replacement
\[
\left( \vec{k} \cdot \vec{q} \right)^2 \rightarrow \kappa^2 q^2 \tag{30}
\]
in (27) will result in a more rapidly oscillatory \( e^{i\omega t} \) with respect to \( \kappa \). We can use the approximated phase [Eq. (26)] with replacement of Eq. (30) in (24).

Before we proceed further, let us introduce a \( \tau \) function
\[
\tau(p, z') = \int_0^{z'} \zeta(z, p) dz, \tag{31}
\]
where the vertical slowness at depth \( z \) is
\[
\zeta(z, p) = \left[ c^{-2}(z) - p^2 \right]^{1/2}. \tag{32}
\]
This function has all the kinematic information we want (Buland and Chapman, 1983). For example, the horizontal distance traveled by a ray with ray parameter \( q \) from depth \( z' \) to depth zero is \( \tau_p |_{p=q} = -\partial \tau / \partial p |_{p=q} \). The second-order derivative of \( \tau \) with respect to \( p \) contains information on the geometrical spreading of the wave front. The \( \tau \) function is the Legendre transform of the travel time function. So the travel time can be easily computed using this function. Using (30) in (27), we obtain
\[
\theta(\vec{k}, \vec{q}, \omega, z') = \frac{1}{2} \tau_{pp} \bigg|_{p=q} \frac{\kappa^2}{\omega^2}. \tag{33}
\]
The subscript " \( \tau_{pp} \) " denotes second-order derivative with respect to the argument \( p \), that is, \( \tau_{pp} = \partial^2 \tau / \partial p^2 \). The derivative of \( \theta \) with respect to depth is
\[
\frac{\partial \theta(\vec{k}, \vec{q}, \omega, z')}{\partial z'} = \frac{1}{2} \tau_{pp} \frac{\kappa^2}{\omega^2}. \tag{34}
\]
The complex phase (24) can be expressed as
\[
\psi(\vec{q}, \vec{x}) \approx -(2\pi)^{-2} \int_0^{t_L} \int \int \int \int e^{i\omega t - i\vec{k} \cdot \vec{r} + i\vec{k} \cdot \vec{x}} \, dv(\vec{k}, z'). \tag{35}
\]
where
\[
a(q, z') = \frac{k^2(z')}{k_z(q, z')}. \tag{36}
\]
5. Coherence Function Between Two Plane Waves

Considering two plane waves with slowness vectors, $\vec{q}_1$ and $\vec{q}_2$, we can express the corresponding complex phases $\psi_1$ and $\psi_2$ at depth zero as:

$$
\psi_1 = \psi(\vec{q}_1, \vec{x}_1) \approx -(2\pi)^{-2} \int_0^L ia_1(z')dz' \int_\kappa e^{i\omega\theta_1(z') - i\vec{\kappa}_1 \cdot \vec{r}_1(z')} d\nu(\vec{\kappa}_1, z'),
$$

(37)

and

$$
\psi_2 = \psi(\vec{q}_2, \vec{x}_2) \approx -(2\pi)^{-2} \int_0^L ia_2(z'')dz'' \int_\kappa e^{i\omega\theta_2(z'') - i\vec{\kappa}_2 \cdot \vec{r}_2(z'')} d\nu(\vec{\kappa}_2, z'').
$$

(38)

The symbols are

$$
a_1(z') = a(q_1, z''), \quad a_2(z'') = a(q_2, z''),
$$

(39)

$$
\theta_1(z') = \theta(\vec{\kappa}_1, \vec{q}_1, \omega, z') = \frac{1}{2} \tau_{pp'}|p = q_1 \frac{\kappa_1^2}{\omega^2},
$$

(40)

$$
\theta_2(z'') = \theta(\vec{\kappa}_2, \vec{q}_2, \omega, z'') = \frac{1}{2} \tau_{pp'}|p = q_2 \frac{\kappa_2^2}{\omega^2},
$$

(41)

and

$$
\vec{r}_1(z') = \vec{r}(\vec{q}_1, z'), \quad \vec{r}_2(z'') = \vec{r}(\vec{q}_2, z'').
$$

(42)

The coherence function between $\psi_1$ and $\psi_2$ is

$$
\langle \psi_1 \psi_2^* \rangle = (2\pi)^{-4} \int_0^L \int_0^L a_1(z')a_2(z'')dz'dz''
$$

$$
\times \iint \int_\kappa \int_\kappa e^{i\omega(\theta_1(z') - \theta_2(z')) + i\vec{\kappa}_1 \cdot \vec{r}_1(z') - \vec{\kappa}_2 \cdot \vec{r}_2(z'')} d\nu(\vec{\kappa}_1, z') d\nu(\vec{\kappa}_2, z''),
$$

(43)

where $\langle \cdot \rangle$ is the ensemble average from multiple realizations of the random medium. For a brief introduction on random variables and random function, and several useful correlation functions and their spectral representations, see Appendix. It can be shown that in the 3-D case (Tatarskii, 1971) that

$$
\langle d\nu(\vec{\kappa}_1, z') d\nu^*(\vec{\kappa}_2, z'') \rangle = (2\pi)^2 W(\vec{\kappa}_1, z', z'') \delta(\vec{\kappa}_1 - \vec{\kappa}_2) d^2\vec{\kappa}_1 d^2\vec{\kappa}_2.
$$

(44)

In the 2-D case, we need replace $2\pi^2$ by $2\pi$ at the right-hand side of Eq. (44). In view of the simple substitution of $d\nu$ by $vd^2\vec{\kappa}$ in Section 4, we see that $W(\vec{\kappa}_1, z', z'')$ is the
correlation function of the two spectral fields \( v(\kappa_1', z') \) and \( v(\kappa_1, z'') \) at depths \( z' \) and \( z'' \), respectively. Applying the coordinate transformation

\[
\eta = z' - z'', \quad \zeta = \frac{z' + z''}{2}
\]  

(45)

in (43) and if the correlation function of the heterogeneity is slowly varying with depth, we can approximate \( W \) as

\[
W(\bar{\kappa}, z', z'') \approx W(\bar{\xi}, \bar{\kappa}, \eta).
\]  

(46)

Equation (43) can be simplified as

\[
\langle \psi_1 \psi_2^* \rangle = (2\pi)^{-2} \int_{0}^{d} d\xi \left\{ a_1(\xi + \eta/2) a_2(\xi - \eta/2) d\xi \right\}
\]

\[
\times \int_{\kappa} \int_{\kappa'} e^{i\theta_1(\xi + \eta/2 - \xi_0(\xi - \eta/2)} |\bar{r}_1(\xi) - \bar{r}_2(\xi)| W(\bar{\xi}, \bar{\kappa}, \eta) d^2 \kappa d^2 \kappa'.
\]  

(47)

Because the correlation function \( W \) decreases rapidly with the vertical separation distance \( |\eta| \), we can extend the integration limit of \( \eta \) from \(-\infty\) to \(+\infty\) without introducing much error. We also make following approximations:

\[
a_1(\xi + \eta/2) \approx a_1(\xi), \quad a_2(\xi - \eta/2) \approx a_2(\xi),
\]  

(48)

\[
\bar{r}_1(\xi + \eta/2) \approx \bar{r}_1(\xi), \quad \bar{r}_2(\xi - \eta/2) \approx \bar{r}_2(\xi).
\]  

(49)

Using Eqs. (48) and (49) in (47), we obtain

\[
\langle \psi_1 \psi_2^* \rangle = (2\pi)^{-2} \int_{0}^{d} d\xi \int_{-\infty}^{+\infty} a_1(\xi) a_2(\xi) d\eta
\]

\[
\times \int_{\kappa} \int_{\kappa'} e^{i\theta_1(\xi + \eta/2 - \xi_0(\xi - \eta/2)} |\bar{r}_1(\xi) - \bar{r}_2(\xi)| W(\bar{\xi}, \bar{\kappa}, \eta) d^2 \kappa d^2 \kappa'.
\]  

(50)

The integral value (50) is nonnegligible only when \( |\eta| \) is small, so we can have the following expansion

\[
\theta_1(\xi + \eta/2) - \theta_2(\xi - \eta/2) \approx \theta_1(\xi) - \theta_2(\xi) + \frac{\theta_1 + \theta_2}{2} \eta,
\]  

(51)

where

\[
\dot{\theta}_1 = \left. \frac{\partial \theta_1(z)}{\partial z} \right|_{z = \xi}, \quad \dot{\theta}_2 = \left. \frac{\partial \theta_2(z)}{\partial z} \right|_{z = \xi}.
\]  

(52)
Taking into account Eq. (51), Eq. (50) can be written as

$$\langle \psi_1 \psi_2^* \rangle = (2\pi)^{-2} \int_0^L d\xi \int \frac{d^2 \vec{k}}{(2\pi)^2} \mathcal{E} \left[ a_1(\xi) a_2(\xi) \int \frac{d^2 \vec{k}}{(2\pi)^2} e^{i\vec{k} \cdot [\vec{r}_2(\xi) - \vec{x}_2] - i\vec{k} \cdot [\vec{r}_1(\xi) - \vec{x}_1]} \right]$$

\hspace{1cm} \times \int_{-\infty}^{+\infty} d\eta e^{i(\eta/2)(\theta_1 + \theta_2)} W(\xi, \vec{k}, \eta).$$

(53)

The depth-dependent power spectrum $P$ and correlation function $W$ are Fourier transform pairs, which can be shown as

$$W(\xi, \vec{k}, \eta) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} P(\xi, \vec{k}) e^{i\xi \eta} d\xi.$$

(54)

Combination of Eqs. (53) and (54) yields

$$\langle \psi_1 \psi_2^* \rangle \approx (2\pi)^{-2} \int_0^L d\xi a_1(\xi) a_2(\xi) \int \frac{d^2 \vec{k}}{(2\pi)^2} e^{i\vec{k} \cdot [\vec{r}_2(\xi) - \vec{x}_2] - i\vec{k} \cdot [\vec{r}_1(\xi) - \vec{x}_1]}$$

\hspace{1cm} \times e^{i(\theta_1 - \theta_2)} P \left[ \xi, \vec{k}, \frac{\omega(\theta_1 + \theta_2)}{2} \right].$$

(55)

We can also derive

$$\langle \psi_1 \psi_2 \rangle \approx -(2\pi)^{-2} \int_0^L d\xi a_1(\xi) a_2(\xi) \int \frac{d^2 \vec{k}}{(2\pi)^2} e^{i\vec{k} \cdot [\vec{r}_2(\xi) - \vec{x}_2] - i\vec{k} \cdot [\vec{r}_1(\xi) - \vec{x}_1]}$$

\hspace{1cm} \times e^{i(\theta_1 - \theta_2)} P \left[ \xi, \vec{k}, \frac{\omega(\theta_1 - \theta_2)}{2} \right].$$

(56)

The Log amplitude $u$ and phase $\phi$ can be expressed by the complex phase

$$u = \frac{\psi + \psi^*}{2} \quad \text{and} \quad \phi = \frac{\psi - \psi^*}{2i}.$$  

(57)

Using this relation to the two plane waves, 1 and 2, we have log $A$ coherence function

$$\langle u_1 u_2 \rangle = \frac{1}{2} \text{Re} \langle \psi_1 \psi_2^* \rangle + \frac{1}{2} \text{Re} \langle \psi_1 \psi_2 \rangle,$$

(58)

and the phase coherence function

$$\langle \phi_1 \phi_2 \rangle = \frac{1}{2} \text{Re} \langle \psi_1 \psi_2^* \rangle - \frac{1}{2} \text{Re} \langle \psi_1 \psi_2 \rangle.$$  

(59)
We can also form the logA-phase coherence function

$$\langle u_1 \phi_2 \rangle = \frac{1}{2} \text{Im}\langle \psi_1 \psi_2 \rangle - \frac{1}{2} \text{Im}\langle \psi_1 \psi_2^* \rangle. \quad (60)$$

If \( \tilde{q}_1 = \tilde{q}_2 \), the two plane waves coming from same direction, we obtain TCFs and they depend only on the spatial lag \( x_1 - x_2 \). Note that these waves are not necessarily vertical incidences as in the Chernov theory. If \( \tilde{x}_1 = \tilde{x}_2 \), the ACFs are obtained for plane waves, \( \tilde{q}_1 \) and \( \tilde{q}_2 \). Because the background velocity profile is depth dependent, these ACFs depend on \( \tilde{q}_1 \) and \( \tilde{q}_2 \) independently, not necessarily the difference \( \tilde{q}_1 - \tilde{q}_2 \). The most general case is JTACF, \( \tilde{x}_1 \neq \tilde{x}_2, \tilde{q}_1 \neq \tilde{q}_2 \).

For 3-D isotropic heterogeneous media, the coherence functions can be simplified using the following identity (Abramowitz and Stegun, 1965)

$$2\pi J_0(\kappa R) = \int_0^{2\pi} e^{-i\kappa \cos \alpha} d\alpha,$$

where \( J_0 \) is the 0th order Bessel function. Therefore, we arrive at

$$\langle \psi_1 \psi_2^* \rangle \approx (2\pi)^{-1} \int_0^L d\xi |a_1(\xi)|a_2(\xi) \times \int_0^\infty \kappa d\kappa J_0[\kappa R(\xi)] e^{i\omega(\tilde{\theta}_1 + \tilde{\theta}_2)/2} P[\xi, \kappa, \omega(\tilde{\theta}_1 + \tilde{\theta}_2)/2],$$

where \( R(\xi) = |r_2(\xi) - \tilde{x}_2 - \tilde{r}_1(\xi) + \tilde{x}_1| \) has an obvious meaning, the transverse distance between two rays at depth \( \xi \), with slownesses \( \tilde{q}_1 \) and \( \tilde{q}_2 \), respectively (see Fig. 1).

Likewise,

$$\langle \psi_1 \psi_2 \rangle \approx -(2\pi)^{-1} \int_0^L d\xi |a_1(\xi)|a_2(\xi) \times \int_0^\infty \kappa d\kappa J_0[\kappa R(\xi)] e^{i\omega(\tilde{\theta}_1 + \tilde{\theta}_2)/2} P[\xi, \kappa, \omega(\tilde{\theta}_1 - \tilde{\theta}_2)/2],$$

For 2-D case, Eqs. (55) and (56) are

$$\langle \psi_1 \psi_2^* \rangle \approx (2\pi)^{-1} \int_0^L d\xi |a_1(\xi)|a_2(\xi) \int_\kappa d\kappa e^{i\kappa(r_2(\xi) - x_2) - i\kappa r_1(\xi) - x_1} \times e^{i\omega(\tilde{\theta}_1 - \tilde{\theta}_2)/2}$$

$$\times P[\xi, \kappa, \omega(\tilde{\theta}_1 + \tilde{\theta}_2)/2], \quad (64)$$

$$\langle \psi_1 \psi_2 \rangle \approx -(2\pi)^{-1} \int_0^L d\xi |a_1(\xi)|a_2(\xi) \int_\kappa d\kappa e^{i\kappa(r_2(\xi) - x_2) - i\kappa r_1(\xi) - x_1}$$

$$\times e^{i\omega(\tilde{\theta}_1 + \tilde{\theta}_2)/2} P[\xi, \kappa, \omega(\tilde{\theta}_1 - \tilde{\theta}_2)/2]. \quad (65)$$
6. COHERENCE FUNCTIONS USING DELTA-CORRELATED ASSUMPTION

The delta-correlated assumption between two depths is often invoked to simplify computation for wave propagation in stochastic media (Tatarskii, 1971; Ishimaru, 1978; Wu and Flatté, 1990)

\[
\langle dv(\vec{k}_1, z') dv^*(\vec{k}_2, z'') \rangle = (2\pi)^m W(\vec{k}_1, z', z'') \delta(\vec{k}_1 - \vec{k}_2) \delta(z' - z'') d\vec{k}_1 d\vec{k}_2,
\]

(66)
in which \( m = 1 \) for 2-D case and \( m = 2 \) for 3-D case. Under this assumption, the vertical wave number of the power spectrum \( P \) is zero, thus Eqs. (55) and (56) reduce to

\[
\langle \psi_1 \psi_2^* \rangle \approx (2\pi)^{-2} \int_0^L d\xi \alpha_1(\xi) \alpha_2(\xi) \int_{\kappa} d^2 \kappa \, e^{i\kappa[z_2(x) - z_1]} e^{i\kappa[z_1(x) - x_1]} p(\xi, \kappa, 0),
\]

(67)
and

\[
\langle \psi_1 \psi_2 \rangle \approx - (2\pi)^{-2} \int_0^L d\xi \alpha_1(\xi) \alpha_2(\xi) \int_{\kappa} d^2 \kappa \, e^{i\kappa[z_2(x) - z_1]} e^{i\kappa[z_1(x) - x_1]} p(\xi, \kappa, 0).
\]

(68)

Various coherence functions according to Eqs. (58)–(60) can be formed as the following:

\[
\langle u_1 u_2 \rangle \approx (2\pi)^{-2} \int_0^L d\xi \alpha_1(\xi) \alpha_2(\xi)
\times \int_{\kappa} e^{i\kappa \cdot [z_2(x) - z_1] - i\kappa \cdot [z_1(x) - x_1]} \sin[\omega \theta_1(\xi)] \sin[\omega \theta_2(\xi)] P(\xi, \kappa, 0) d^2 \kappa,
\]

(69)

\[
\langle \phi_1 \phi_2 \rangle \approx (2\pi)^{-2} \int_0^L d\xi \alpha_1(\xi) \alpha_2(\xi)
\times \int_{\kappa} e^{i\kappa \cdot [z_2(x) - z_1] - i\kappa \cdot [z_1(x) - x_1]} \cos[\omega \theta_1(\xi)] \cos[\omega \theta_2(\xi)] P(\xi, \kappa, 0) d^2 \kappa,
\]

(70)

\[
\langle u_1 \phi_2 \rangle \approx - (2\pi)^{-2} \int_0^L d\xi \alpha_1(\xi) \alpha_2(\xi)
\times \int_{\kappa} e^{i\kappa \cdot [z_2(x) - z_1] - i\kappa \cdot [z_1(x) - x_1]} \sin[\omega \theta_1(\xi)] \cos[\omega \theta_2(\xi)] P(\xi, \kappa, 0) d^2 \kappa.
\]

(71)

Using identity (61), we can explicitly obtain three types of coherence functions in 3-D:
\[ \langle u_1 u_2 \rangle \approx (2\pi)^{-1} \int_0^L d\xi \int_0^\infty J_0(\kappa R(\xi)) \sin[\omega \theta_1(\xi)] \sin[\omega \theta_2(\xi)] P(\xi, \kappa, 0) d\kappa \]  
(72)

\[ \langle \phi_1 \phi_2 \rangle \approx (2\pi)^{-1} \int_0^L d\xi \int_0^\infty J_0(\kappa R(\xi)) \cos[\omega \theta_1(\xi)] \cos[\omega \theta_2(\xi)] P(\xi, \kappa, 0) d\kappa \]  
(73)

\[ \langle u_1 \phi_2 \rangle \approx -(2\pi)^{-1} \int_0^L d\xi \int_0^\infty J_0(\kappa R(\xi)) \sin[\omega \theta_1(\xi)] \cos[\omega \theta_2(\xi)] P(\xi, \kappa, 0) d\kappa. \]  
(74)

### 7. Coherence Functions in a Constant Background Medium

To compare our results with those in Wu and Flatté (1990), we assume that the background velocity model is homogeneous and the incident angle is small. Under these conditions, the \( \tau \) function can be approximated as

\[ \tau(\xi, p) = \int_0^\infty [e^{-2} - p^2]^{1/2} d\zeta \approx e^{-1} \left( 1 - \frac{1}{2} p^2 e^2 \right) \zeta. \]  
(75)

Phase functions \( \theta_1 \) and \( \theta_2 \) can be explicitly solved

\[ \omega \theta_1 = -\frac{1}{2} c \frac{\kappa^2}{\omega} \frac{\xi}{2} = -\frac{\xi}{2k} \kappa^2. \]  
(76)

The amplitudes can be approximated as

\[ a_1(\xi) a_2(\xi) \approx k^2. \]  
(77)

Substituting Eqs. (75)–(77) into Eqs. (72)–(74), the generalized results in this chapter reduce to those contained in Wu and Flatté (1990) for a homogeneous background medium.

### 8. Numerical Examples

To demonstrate the validity of the theory proposed in this chapter, we compared the coherence functions from numerical simulations and theoretical predictions. We constructed a 2-D random model (Fig. 2a) with a background velocity profile same as the IASP91 model (Fig. 2b; Kennett and Engdahl, 1991). Two random layers are superimposed on the background model. The top layer is from 0 to 120 km in depth and it has a Gaussian correlation function with correlation length 10 km in both horizontal and vertical directions, and rms 1% of the background velocity for those random velocity perturbations. The bottom layer extending from 120 to 310 km depth also has a Gaussian
correlation function with correlation length 20 km in both directions and rms 1% of the background velocity. For both layers, we constrain the velocity perturbations not exceeding ±3% of the background velocity. Below 310 km depth, the medium is homogeneous and has no random perturbations and the velocity is same as the one at the 310 km depth. We used a full-wave finite difference code (Xie, 1988) to simulate the acoustic plane wave propagation in the random medium. The spectral amplitude and phase fluctuations at a given frequency are extracted from the waveforms recorded at the surface (Zheng and Wu, 2005). Then coherence functions are formed using those fluctuations and averaged over an ensemble of 100 different stochastic model realizations. The logA (Fig. 3) and phase (Fig. 4) coherence functions at 0.5 Hz from the numerical simulations are well predicted by our formula for four different incidence geometries between two plane waves. The incidence angle is measured at the surface for consistency. The logA coherence functions in Fig. 3 are sensitive to the angular separation between two incident plane waves. However, phase coherence functions are relatively simple (Fig. 4) in shape. Another salient feature is that the maximum amplitude of the coherence function (for both logA and phase) is decreasing with increasing angular separation between the two plane waves, which is expected. It is also interesting to notice

Fig. 2. (a) Random velocity model used in the 2-D numerical modeling; (b) 1-D iasp91 background velocity model (solid line) and a vertical velocity profile (dash line) at location $X = 600$ km, to indicate the randomness of the velocity fluctuation.
that there are several velocity discontinuities in the iasp91 background model and this seems to pose difficulty to using the WKBJ Green’s function as discontinuities can produce reflected waves. However, because we are using plane wave incidence, those discontinuities affect all receivers in a similar fashion and such effect is removed during coherence function formation.

9. Validity of the Delta-Correlated Assumption

In Section 6, we see that the delta-correlated assumption significantly simplifies the mathematical derivation of the coherence functions. This assumption is equivalent to certain conditions under which the result of the coherence function is same to that obtained as if the medium is delta-correlated along the depth direction. The condition is easily found to be \( \kappa^2 \eta / 2k \ll 1 \) in view of Eqs. (53) and (76) for a homogeneous background medium. We assume that \( \Lambda \) is the smallest correlation length at which \( W(\xi, \kappa, \eta) \) is only slightly different from zero. We also assume that \( P(\xi, \kappa) \sim 0 \) if \( \kappa > \kappa_m = 2\pi/\ell_0 \), with \( \ell_0 \) being the inner scale of the random medium. Therefore, if condition \( \lambda \Lambda \ll \ell_0 \) is satisfied, the delta-correlated assumption will hold. Here \( \lambda \) is the wavelength. Of course, we generally further require \( \ell_0 \ll L \). To assess the error

Fig. 3. log A coherence functions \( u_1 u_2 \) for two plane waves with incidence angles, 0° and 0° (a), 0° and 5° (b), 0° and 10° (c), and 0° and 15° (d). Numerical results are indicated as dash lines and the theoretical predictions are the solid lines.
introduced by this assumption, we need numerical verification for a given random model. For a homogeneous background medium, Zheng et al. (2007) numerically found that the delta-correlated assumption will cause little error if the layer thickness of the random medium is large for short wave propagation. Using the model presented in Fig. 2, we computed coherence functions with (Section 6) and without (Section 5) the delta-correlated assumption and we found the results are basically the same. The delta-correlated assumption will not significantly simplify the coherence function calculation. However, it does provide convenience for us to invert for the depth-dependent random spectrum. With the delta-correlated assumption, we can specify the unknowns easily without concerning the vertical wave number in the spectrum $P$.

10. Discussions and Conclusions

Theory of the coherence function for both log $A$ and phase in a depth-dependent background velocity model is important to draw correct inference on the random medium property. In the past, coherence functions have been theorized in the context of using a constant background medium, thus a simple Green’s function. In this chapter, the natural spectral representation of the Green’s function in the wave number domain allows us to extend the previous theory to a scenario where the background medium is depth-dependent.
The agreement between the numerical simulation and the theoretical prediction shows the correctness of the theory and the potential to apply the theory to real seismic data. We formulated the coherence function both in 3-D case and in 2-D case, with and without the delta-correlated assumption. The delta-correlation does not lead to discernable differences for the model we used to do the simulation. However, with the delta-correlated assumption, the parameterization for the inverse problem can be easier.

The previous theory on the coherence function $C$ for the log $A$ or for the phase depends on the spatial lag between stations, and the angle separation between the two plane waves, $C = C(\vec{r}_2 - \vec{r}_1, q_2 - q_1)$. However, this is valid only when the slowness vectors of the two plane waves are close, that is $\vec{q}_1 \approx \vec{q}_2$. This condition cannot always be true and it also limits the depth resolution of the coherence function, resulting the spectral smearing along the depth direction. Our current theory is capable of dealing with much larger angular separation between the two plane waves and the mathematical formulation for the coherence function depends on $\vec{q}_1$ and $\vec{q}_2$ individually, and it takes a function form of $C = C(\vec{x}_2 - \vec{x}_1, q_2, q_1)$.

Acknowledgments

Comments from Professor Haruo Sato, Dr. Tatsuhiko Saito, and an anonymous reviewer have greatly improved the clarity of the chapter. We also thank Dr. Xiao-bi Xie for reading our manuscript and discussions. Y. Zheng is also grateful to Professor Thorne Lay for his encouragement. We thank the W.M. Keck Foundation for facility support. This work is funded in part by DOE/Basic Sciences, NSF EAR0635570, and the Wavelet Transform on Propagation and Imaging Consortium/UCSC. This is Contribution Number 495 of CSIDE, IGPP, University of California, Santa Cruz.
The theory of transmission fluctuations in random media

Let \( B(\mathbf{r}_1 - \mathbf{r}_2) = \langle f(\mathbf{r}_1)f(\mathbf{r}_2) \rangle \). Symbol \( \langle \cdot \rangle \) is used to denote the ensemble average. By stationary random medium we mean that the statistic does not change by a translation \( \mathbf{d} \), for example, \( B(\mathbf{r}_1, \mathbf{r}_2) = B(\mathbf{r}_1 + \mathbf{d}, \mathbf{r}_2 + \mathbf{d}) \). If the correlation function only depends on the Euclidean distance between \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), the medium is called isotropic, that is, \( B(\mathbf{r}_1, \mathbf{r}_2) = B(|\mathbf{r}_1 - \mathbf{r}_2|) \). To obtain the isotropic spectrum \( P(\kappa) \), we use a Fourier transform of the correlation function, that is, \( P(\kappa) = \int B(r)e^{-i\kappa \mathbf{r}} dr \). In numerical studies, it is practical to produce realizations of random media of certain correlation function and this can be done in the spectral domain (Shapiro and Kneib, 1993). We first assign the spectral amplitude at each wave number then generate random phases in range \([-\pi, \pi]\).

Of course, the complex conjugate symmetry has to be used if a real random medium is desired. We list two most common correlation functions and their spectra. The Gaussian correlation function reads \( B(r) = \varepsilon^2 \exp\left(-r^2/r_0^2\right) \). \( \varepsilon^2 \) is the perturbation strength and \( r_0 \) is the correlation length. Its Fourier transform in \( N \) dimension is \( P(\kappa) = \varepsilon^2 N^{N/2} \exp\left(-\kappa^2 r_0^2/4\right) \). The exponential correlation function can be expressed as \( B(r) = \varepsilon^2 e^{-r/r_0} \). Its Fourier transforms are \( P(\kappa) = 2\pi \varepsilon^2 r_0/(1 + \kappa^2 r_0^2) \) in the 1-D case and \( P(\kappa) = 8\pi \varepsilon^2 r_0^2/(1 + \kappa^2 r_0^2)^2 \) in 3-D case. These results can be easily generalized to anisotropic cases, in which the correlation lengths in different directions (with correlation lengths, \( r_{0x}, r_{0y}, \) and \( r_{0z} \), in \( x, y, \) and \( z \) directions, respectively) can be different, \( r_{0x} \neq r_{0y} \neq r_{0z} \). The simplest way is to do the following replacement, \( \kappa^2 \sim \kappa_x^2 + \kappa_y^2 + \kappa_z^2; r_0^2 \sim r_{0x}r_{0y} \) in 2-D and \( r_0^2 \sim r_{0x}r_{0y}r_{0z} \) in 3-D case.

References


