Inverting the power spectrum for a heterogeneous medium

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SUMMARY
The delta-correlated assumption of the random medium is of considerable significance in statistical wave propagation problems. Assuming the Rytov and parabolic approximations, this work completed an exact derivation of the transverse coherence functions without using the delta-correlated approximation. The error introduced by this approximation is shown to be negligible when the source–receiver distance is large. Our new formulation shows that the TCF does have limited vertical spectral resolution, in contrast to previous results. We define a new observational variable whose Fourier transform directly yields the power spectral density function of the random medium. This enables a simultaneous determination of the scale and statistical distribution of heterogeneities without assuming the power spectral function a priori.

Key words: coherence function, Rytov approximation, scattering, velocity heterogeneity.

1 INTRODUCTION
The delta-correlated assumption of the random medium in the vertical direction has been used extensively in statistical wave propagation problems (e.g. Tatarskii 1971; Ishimaru 1978; Wu & Flatté 1990) to mathematically simplify the calculation of transverse coherence functions (TCFs), namely, the cross-coherence functions of phase and log-amplitude fluctuations. Such functions are important in determining heterogeneity scales and perturbation strengths (Flatté & Wu 1988). However, the validity of the delta-correlated assumption has rarely been examined rigorously in the literature. In this paper, we will present a rigorous mathematical derivation on this approximation and its application in the study of TCFs. To illustrate the problem, we first present a brief overview of the Rytov and parabolic approximations, then an exact TCF formulation without the delta-correlated assumption including reinterpretation of the meaning and validity of this assumption, and finally, numerical examples are presented for inverting the spectrum of a stationary random medium.

2 RYTOV AND PARABOLIC APPROXIMATIONS
The Rytov approximated solution to the wave equation is the first-order term in a series arising from the method of smooth perturbation (e.g. Tatarskii 1971; Rytov et al. 1989). It is a single-scattering theory and does not account for the multiple scattering, which limits it to small velocity perturbations of the medium. The perturbation method (e.g. Wu 1989) is among the most popular theoretical tools for study of wave propagation in random medium. In this method, the random velocity field is decomposed into a constant background velocity plus the remaining perturbation. The monochromatic wave equation for the wave motion $U$ for acoustics in such a medium with constant density reads,

$$[\nabla^2 + k^2]U = -2k^2 \delta n(\vec{r}) U,$$

where $k = \omega/c_0$; $\nabla$ and $\omega$ are Laplacian differential operator and the angular frequency; $\delta n(\vec{r}) = \frac{1}{2}(c_0^2/c^2 - 1)$ is the dimensionless velocity fluctuation. Let $U_0$ be the background wavefield without the velocity fluctuation, that is,

$$[\nabla^2 + k^2]U_0 = 0.$$

The background wavefield can be equated to the perturbed wavefield by

$$U = U_0 e^{i\psi}.$$

For the sake of brevity, we dropped the explicit dependent of the functions (e.g. $U$, $U_0$ and $\Psi$) on the angular frequency $\omega$. Eqs (1), (2) and (3) together yield an integral equation,

$$\psi(\vec{r}) = \frac{1}{U_0(\vec{r})} \int F(\vec{r}, \vec{r}') U_0(\vec{r}') [2k^2 \delta n(\vec{r}') + |\nabla \psi(\vec{r}')|^2] d^3\vec{r}'.$$
The log-amplitude (log-A hereafter) fluctuation can be easily recognized as (see previous section)

\[ \psi = \frac{2k^2}{\langle U(\vec{r})^2 \rangle} \int_{0}^{L} G(\vec{r}, \vec{r}') U_0(\vec{r}') \delta n(\vec{r}') d^3\vec{r}'. \]  (5)

It is recognized that the Rytov approximation is superior to the Born approximation in that the Rytov approximation only requires the smoothness of \( \psi(\vec{r}) \) while the Born approximation additionally needs \( |\psi(\vec{r})| \gg 1 \). The wavefield can be expressed in terms of amplitude and phase, \( U = Ae^{\phi} \), \( U_0 = Ae^{\phi_0} \) and \( \psi = \ln(A/A_0) + i(\phi - \phi_0) \). Physically, the Born approximation is valid only in a weak scattering regime and for short propagation distances.

Short-wave propagation in random medium is one of the most extensively studied subjects by many people, partly because it can be approached with simpler theory. When the wavelength \( \lambda \) is large compared to \( a \) the length scale of heterogeneity, the scattering is isotropic (i.e. scattered wave amplitudes are the same in all directions), but when \( \lambda \) is small, the scattered energy mainly concentrates in a forward cone with angular span \( \sim 1/ka \), \( k \) being the wavenumber (e.g. Sato & Fehler 1998). Under the small-angle approximation suitable for the latter situation, TCFs for 2- and 3-D media are the same. We will use the parabolic approximation to simplify eq. (5) in the 2-D case with a vertically incident plane wave,

\[ \psi(x, L) \approx 2k^2 \int_{0}^{L} d'z' \int_{-\infty}^{+\infty} dx G(x, L; x', z') \delta n(x', z') \exp[-ik(L - z')]. \]  (6)

The Green’s function in the background medium reads,

\[ G(x, L; \vec{x}', z') = \frac{i}{4} H_0(kR), \quad R = \sqrt{(x - x')^2 + (L - z')^2}, \]

where \( H_0(kR) \) denotes the zero-order Hankel function of the first kind. When \( kR \) is large, the asymptotic representation of \( G \) becomes,

\[ G(x, L; \vec{x}', z') \approx \frac{i}{4} e^{-i\pi/4} \sqrt{\frac{2}{\pi k(L - z')}} \exp \left[ i k(L - z') + \frac{ik(x - x')^2}{2(L - z')} \right]. \]  (7)

Plugging (7) into eq. (6),

\[ \psi(x, L) \approx \frac{ik^2}{2} e^{-i\pi/4} \int_{0}^{L} d'z' \int_{-\infty}^{+\infty} dx \delta n(x', z') \frac{2}{\pi k(L - z')} \exp \left[ i k(L - z') + \frac{ik(x - x')^2}{2(L - z')} \right]. \]  (8)

Make use of the following spectral representation of medium fluctuation

\[ \delta n(x, z) = \frac{1}{2\pi} \int e^{i\kappa z} d\nu(\kappa, z), \]

the convolution of eq. (8) becomes multiplication in \( \kappa \) domain,

\[ \psi(x, L) = \frac{1}{\pi} \int_{0}^{L} d'z' \int e^{i\kappa z} H(\kappa, L - z') d\nu(\kappa, z'), \]  (10)

where

\[ H(\kappa, L - z') = \frac{ik}{2} \exp \left[ -i \frac{(L - z')\kappa^2}{2k} \right]. \]

Strictly speaking, an infinite random medium does not have a spectrum in the Fourier sense, because the existence of a Fourier representation requires \( \delta n(x, z) \) be absolute integrable with respect to variable \( x \). Eq. (9) should be understood as a stochastic Fourier–Stieltjes integral of spectral representation for a random medium (e.g. Yaglom 1962; Tatarskii 1971) with \( d\nu(\kappa, z') \) being a spectrum at wavenumber \( \kappa \). For computational purposes, the conventional Fourier spectrum can be used in eq. (9) by assuming periodicity of a finite dimensional random medium.

## 3 Delta-Correlation and TCF Theory

The log-amplitude (log-A hereafter) fluctuation can be easily recognized as (see previous section)

\[ u(x, L) = \frac{1}{2} (\psi + \psi^*) = \frac{1}{\pi} \int_{0}^{L} d'z' \int e^{i\kappa z} H(\kappa, L - z') d\nu(\kappa, z'), \]  (11)

where \( H(\kappa, L - z') \) is the real part of \( H(\kappa, L - z') \) and \( * \) denotes complex conjugate.

The coherence of log-A is,

\[ \langle uu \rangle = \frac{1}{\pi^2} \int_{0}^{L} d'z' \int_{0}^{L} d''z'' \int d\kappa d\nu e^{i\kappa z'} H(\kappa, L - z') H^*(\kappa, L - z'') \langle d\nu(\kappa, z') d\nu(\kappa, z'') \rangle, \]  (12)

where \( \langle \rangle \) implies ensemble average. The delta-correlation assumption of the medium at different depths states that \( W(\kappa, z' - z'') = \langle d\nu(\kappa, z') d\nu^*(\kappa, z'') \rangle = W(\kappa, 0) \delta(z' - z'') \).  (13)
With this approximation, eq. (12) can be calculated in an effortless way. Note that $W(\kappa, \eta)$ is an even function about $\eta$. Our purpose is to examine its validity and abandon this approximation from now on. Apply trigonometrical identities to eq. (12), we get,

$$
H_r(\kappa, L - z)H_r(\kappa, L - z') = \frac{k^2}{4\pi^2} \left[ \cos \left( \frac{\xi^2}{2k} \right) - \cos \left( \frac{L - \xi}{k} \right) \right] W(\kappa, \eta).
$$

(14)

Substituting identity (14) into eq. (12) and making use of coordinate transformation (Fig. 1)

$$
\xi = (z' + z'')/2, \quad \eta = z' - z''.
$$

Eq. (12) becomes,

$$
\langle uu \rangle = \frac{k^2}{4\pi^2} \int_0^L d\xi \int d\eta \int d\kappa e^{i\kappa z} \left[ \cos \left( \frac{\eta k^2}{2k} \right) - \cos \left( \frac{L - \xi}{k} \right) \right] W(\kappa, \eta)
$$

$$
= \frac{k^2}{4\pi^2} \int_0^L d\eta \int d\kappa e^{i\kappa z} \frac{\eta k^2}{k} W(\kappa, \eta) - \frac{k^2}{4\pi^2} \int \frac{L}{0} d\xi \int d\eta \int d\kappa e^{i\kappa z} \cos \left( \frac{L - \xi}{k} \right) W(\kappa, \eta).
$$

Using the symmetry property of the coordinate transform and noticing the dependence of $\xi$ on $\eta$ (Fig. 1), the 2-D integration can be carried out using 1-D integration.

$$
\langle uu \rangle = \frac{k^2}{4\pi^2} \int d\kappa e^{i\kappa z} \frac{\eta k^2}{k} W(\kappa, \eta)(L - \eta) d\eta - \frac{k^2}{4\pi^2} \int \int d\eta \int d\kappa e^{i\kappa z} \cos \left( \frac{L - \xi}{k} \right) W(\kappa, \eta)
$$

$$
= \frac{k^2}{4\pi^2} \int d\kappa e^{i\kappa z} \left[ \sin \left( \frac{\eta k^2}{2k} \right) - \sin \left( \frac{2L - \xi}{2k} \right) \right] W(\kappa, \eta).
$$

Assume $L > \xi$ for $|z| > \xi$, $W(\kappa, z) \approx 0$, we have

$$
I_2 = \frac{k^3}{4\pi^2} \int d\kappa e^{i\kappa z} \sin \left( \frac{\eta k^2}{2k} \right) W(\kappa, \eta) d\eta,
$$

$$
I_2 = \frac{k^3}{4\pi^2} \int d\kappa e^{i\kappa z} \frac{\eta k^2}{k} \cos \left( \frac{L \kappa^2}{k} \right) W(\kappa, \eta) d\eta - \frac{k^3}{4\pi^2} \int d\kappa e^{i\kappa z} \cos \left( \frac{\eta k^2}{2k} \right) W(\kappa, \eta) d\eta.
$$

The coherence function $W(\kappa, \eta)$ and the random medium power spectrum density function (PSDF)$P(\kappa, \eta)$ in wavenumber domain can be linked by a Fourier transform (e.g. Tatarskii 1971; Wu & Flatté 1990),

$$
P(\kappa, \eta) \sim \int W(\kappa, \eta) e^{i\kappa z} d\eta.
$$

Using this well-known property, $I_1$ and $I_2$ can be further simplified,

$$
I_2 = \frac{k^3}{8\pi^2} \int d\kappa e^{i\kappa z} \sin \left( \frac{\eta k^2}{2k} \right) P(\kappa, \eta) W(\kappa, \eta) d\eta + \frac{k^3}{2\pi} \int d\kappa e^{i\kappa z} \cos \left( \frac{L \kappa^2}{2k} \right) \int_0^\infty \sin \left( \frac{\eta k^2}{2k} \right) W(\kappa, \eta) d\eta
$$

$$
I_1 = \frac{k^2}{8\pi^2} \int d\kappa e^{i\kappa z} \left[ \frac{\eta k^2}{k} e^{i\kappa z} d\kappa - \frac{k^3}{2\pi^2} \int d\kappa e^{i\kappa z} \int_0^\infty \eta \cos \left( \frac{\eta k^2}{2k} \right) W(\kappa, \eta) d\eta.
$$

The TCF of the log-A fluctuation is $\langle uu \rangle = I_1 + I_2$; that for phase can be easily found to be $\langle \phi \psi \rangle = I_1 - I_2$.

Under the delta-correlated assumption (13), we arrive at,

$$
\langle uu \rangle = \frac{k^2}{8\pi^2} \int d\kappa e^{i\kappa z} P(\kappa, \eta) - \frac{k^3}{8\pi^2} \left[ \int d\kappa e^{i\kappa z} \sin \left( \frac{\eta k^2}{2k} \right) P(\kappa, \eta) \right].
$$

(15)
Figure 2. Exact (short dash) and delta-correlation approximated (solid) TCFs, together with the error term (long dash) as a function of the correlation lag distance for three different propagation distances \( L \) (a) \( L = 14a \), (b) \( L = 10a \) and (c) \( L = 6a \). Isotropic exponential random medium with scale length \( a_x = a_z = a = 1 \text{ km} \) is used in calculation. The wavelength \( \lambda \) is \( \lambda = a/2 \).

\[
\langle \phi \phi \rangle = \frac{Lk^2}{8\pi^2} \int d\kappa e^{i\kappa x} P\left(\kappa, \frac{k^2}{2k}\right) + \frac{k^3}{8\pi^2} \int d\kappa e^{i\kappa x} \sin\left(\frac{L\kappa^2}{k^2}\right) P\left(\kappa, \frac{k^2}{2k}\right).
\]  

(16)

The second term in the right hand side of \( I_1 \) (or \( I_2 \)) is the error introduced by delta-correlation assumption. The error diminishes monotonically for log-A and phase fluctuations as the propagation distance \( L \) increases (Fig. 2). However, when keeping the propagation distance fixed \( L = 6a \), the errors for phase fluctuations are little influenced by the wavelength (Fig. 3). In both cases, the log-A TCFs broaden with increasing wavelengths or increasing propagation distances. This demonstrates that heterogeneities having size of the Fresnel zone \( \sqrt{L} \) are of great importance in log-A fluctuations. However, the coherence function for the phase fluctuation is not sensitive to either \( L \) or \( \lambda \), but to the actual length scale of the heterogeneity. These findings are in agreement with those in Tatarskii (1971).

Add eqs (15) and (16), we get,

\[
4\pi^2\langle uu \rangle + \langle \phi \phi \rangle = \int P\left(\kappa, \frac{k^2}{2k}\right) e^{i\kappa x} d\kappa.
\]  

(17)

Based on eq. (17), it is interesting to see,

\[
P\left(\kappa, \frac{k^2}{2k}\right) = \frac{2\pi}{Lk^2} \int \langle uu \rangle + \langle \phi \phi \rangle e^{-i\kappa x} dx.
\]  

(18)

In fact, with no delta-correlation approximation, we expect,

\[
P\left(\kappa, \frac{k^2}{2k}\right) = \frac{2\pi}{Lk^2} \int \langle uu \rangle + \langle \phi \phi \rangle e^{-i\kappa x} dx + \frac{2}{L} \int_0^\infty \eta \cos\left(\frac{\eta k^2}{2k}\right) W(\kappa, \eta) d\eta.
\]  

(19)

These results should be valid if the random medium is smooth (Rytov approximation) and the wavelength is small compared to the heterogeneity size (parabolic approximation). In this theory, with no delta-correlation assumption, the criterion for the minimum propagation distance is quantified as well \( L > \ell \). However, in both exact and delta-correlated cases, the second argument (vertical component) in the inverted spectrum \( P \) is not zero. This is different from the work by Flatté & Wu (1988) and Wu & Flatté (1990) in which the vertical wavenumber is zero.

It is clear that TCFs only yield partial spectral information of the heterogeneities and the deterministic model cannot be resolved by these functions. However, they are sufficient to constrain the sizes of the heterogeneities, which are important in studying the Earth’s internal dynamics. Eqs (18) and (19) are the key results presented in this paper. They allow direct inversion for the power spectrum of the random medium. In an anisotropic random medium (i.e. the power spectrum of the scatters varies with angle), eq. (18) yields partial spectral information.
of the random media; however, in isotropic random medium, it provides the full PSDF of the random medium. Its shape provides important information on the statistical distribution of the heterogeneities. For a stationary random medium, eq. (18) demonstrates the trade-off between scattering strength \( P \) and the random layer thickness \( L \) because the inverted quantity based on data is \( L \cdot P(\kappa) \), resolving either of them (\( L \) or \( P \)) needs the information of the other. The second term in eq. (19) is proportional to the inverse of the propagation distance. Given \( \kappa \), the integral is a fixed quantity. When \( L \) is large, this term can be dropped. If the term cannot be dropped, recognizing \( P \) and \( W \) are Fourier transform pairs, some iterative method can be employed to solve eq. (19). Next, we use a simple example to show the theory is valid.

### 4 Numerical Example of Spectral Inversion

In the example, we use eq. (18) to do the inversion. We consider a model with dimension of 40 km-by-40 km with a background velocity of 2.5 km s\(^{-1}\). A vertically propagating plane wave impinges at the bottom of the model at \( z = 0 \) km and 10 receiver lines are placed at depths \( z_i = i \times 2.42 \) km. We synthesize a random medium with a Gaussian PSDF with root-mean-square velocity perturbations of 1 per cent and characteristic scale \( a = 0.5 \) km. The upgoing plane wave has a Ricker source time function of central frequency \( f_0 = 4.5 \) Hz. The log-A and phase coherence functions are formed at different propagation distances using synthetic seismograms computed by a FD method (Xie 1988). The FD method has fourth- and second-order accuracy in spatial and temporal derivatives. Fig. 4 displays power spectra estimated from log-A/phase measurements at various depths (i.e. propagation distances). Log-A and phase coherence functions are formed using the method described by Zheng & Wu (2005). The error, \( \delta \), introduced by the delta-correlation assumption also can be studied quantitatively and accurately by this method, which is different with the error estimation approach taken by Tatarskii (1971). Past work (e.g. Aki 1973; Capon 1974; Capon & Bertussens 1974; Powell & Meltzer 1984; Flatté & Wu 1988; Flatté et al. 1991) involved parametrical random medium
Figure 4. Inverted power spectra of the random media at different propagation distances (dash lines) and the true spectrum (solid black line). Because of the trade-off between the power spectral amplitude and layer thickness, the spectral amplitude in this diagram only has relative meaning.

spectral assumptions (e.g. Gaussian, exponential, etc.) with the appropriate parameters being fitted to the observed log-A/phase coherence functions.

5 CONCLUSIONS

The TCF theory has been derived in a mathematical rigorous way without using the delta-correlation approximation. We found heterogeneities of size $\sqrt{\lambda L}$ are of importance in determining the log-A fluctuation functions and the phase coherence function is sensitive to the actual scale lengths of heterogeneities. Our new formulation shows the TCF does have limited vertical spectral resolution, which is different from previous results (i.e. vertical wavenumber be 0). We defined a new observational variable whose Fourier transform directly yields the PSDF of the random medium. This can be used to determine the scale and statistical distribution of the heterogeneities with minimal model dependence. Our new approach does not require any a priori assumptions on the power spectrum distribution of the random medium.

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REFERENCES


