Target-oriented beamlet migration based on Gabor-Daubechies frame decomposition

Ling Chen¹, Ru-Shan Wu², and Yong Chen³

ABSTRACT
We develop beamlet propagation and imaging using Gabor-Daubechies (G-D) frame decomposition based on local perturbation theory and apply it to target-oriented prestack depth migration. The method is formulated with local background velocities and local perturbations in wavefield extrapolation. The localized propagators and phase-correction operators are obtained analytically or semianalytically by one-way operator decomposition and screen approximation in the beamlet and space-beamlet mixed domain. Beamlet wavefields have superior localization properties in both local space and direction (wavenumber) over Gaussian beams in the sense that localizations are not limited within short propagation distances in either homogeneous or heterogeneous media. Comparisons of the prestack depth-migrated images for the 2D SEG-EAGE salt model and the Marmousi model indicate that, for seismic-wave propagation and imaging in complex structural environments, the G-D beamlet propagator has higher accuracy and better wide-angle properties than do global propagators. Target-oriented prestack G-D beamlet migration is performed by means of local-angle-domain imaging and controlled superposition of common-dip-angle images based on the local directivity features of the target structures. This considers the spatial and direction localizations of beamlets. As a numerical example, we process the 2D SEG-EAGE salt-model prestack data. The results show that the proposed migration method has considerable advantages in suppressing noise and enhancing structural features. Image quality for subsalt structures, especially for steep faults, is improved through structure-based superposition of common-angle images. This demonstrates the potential and capability of beamlet migration in target-oriented seismic imaging.

INTRODUCTION
Various techniques and methods for wavefield decomposition, propagation, and migration/imaging have been developed and applied to seismic exploration, including the Kirchhoff high-frequency asymptotic method (Schneider, 1978; Gray and May, 1994; Audebert et al., 1997), the wave-equation phase-shift method (Gazdag, 1978; Stolt, 1978), the phase-screen method (Stoffa et al., 1990; Wu and Huang, 1992), and the generalized screen propagator (GSP) method (de Hoop et al., 2000; Xie et al., 2000; Jin et al., 2002; Wu, 2003). In these methods, the wavefield is expanded by sets of basic functions such as spatial Fourier harmonics, modes, and spatial Green’s functions. The common factor in these basic functions is their global nature in either the space or the wavenumber domain. Basic functions in the wavenumber domain are plane waves, which have the best localization in direction but no spatial localization. Basic functions in the space domain are point sources (delta functions), which have the best localization in space but have no direction localization. In highly heterogeneous media, the global nature of these two kinds of basic functions creates some difficulties in correcting for local heterogeneities, which is necessary to improve image quality and to extract information in the target area.

To overcome this fundamental limitation caused by the global nature of these propagators, efforts have been made to investigate and develop wavefield decomposition and extrapolation methods with localization in both space and direction. High-frequency, ray-based beam-summation methods, such as the complex source-generated beam and the Gaussian
beam methods (Deschamps, 1971; Červený et al., 1982; Felsen et al., 1991; Hill, 1990, 2001), have been developed to solve the caustic problem encountered in the traditional geometric ray method and, more importantly, to achieve wavefield extrapolation in a localized way. High-frequency asymptotics and paraxial approximation, however, inevitably limit the accuracy of these methods and their applicability to complex media. Steinberg (1993) and Steinberg and Birman (1995) derive a wave-equation-based localized phase-space propagator using windowed Fourier transforms (WFTs) and a perturbation approach. The localized propagator depends mainly on the local heterogeneity of the medium and is much easier to construct with high accuracy than are global propagators. The perfect WFT decomposition and reconstruction, however, is formidable expensive, so the method is difficult to adopt for practical use. In addition, it is formulated with a global perturbation to the partial differential wave equation, which, for strong lateral variations, may result in large perturbations and might produce significant errors in wavefield propagation. In window screen-propagator methods (Wu and Jin, 1997; Jin and Wu, 1999), local background velocities and local perturbations are introduced through WFT. Although this provides a possible way to improve the accuracy of wavefield extrapolation, the broadly overlapping windows in WFT and the empirical interpolations employed limit the utility of this method.

Advanced mathematical techniques, especially the newly developed wavelet transform and the more general frame theory (Daubechies, 1990, 1992; Mallat, 1998), provide a solid foundation for the development of localized propagators. Wu et al. (2000) and Wu and Chen (2001) propose a wave-equation-based wavefield extrapolation formulation based on local perturbation theory. They derive analytically the localized propagators by using the Gabor-Daubechies (G-D) frame (Appendix A) to decompose and propagate the wavefield in the beamlet domain. The basic functions (vectors) of the G-D frame decomposition are Gaussian enveloped (Gaussian windowed) harmonic wavefields. Applying G-D frame decomposition to the wavefield in a local homogeneous region is like decomposing the wavefield into many small Gaussian beams (beamlets) at different window locations and with different beam directions. The G-D frame decomposition of wavefields is redundant, as measured quantitatively by the redundancy ratio $R$ (see Appendix A) — a condition for stable reconstruction and hence somewhat more expensive than orthogonal decomposition schemes. The spatially and directionally confined frame vectors, however, result in many desirable features of the G-D beamlet wavefield extrapolation method. With local background velocities and local perturbations adopted in each extrapolation step, the G-D beamlet propagator propagates the beamlets step by step in depth, accounting for both individual beamlet propagation and cross-coupling between beamlets through local heterogeneities during propagation.

The G-D beamlet propagator is more flexible and has better wide-angle properties than do traditional global propagators, as manifested by numeric results from forward modeling and poststack migration (Wu and Chen, 2001, 2002a). Moreover, the localization properties in both space and direction (wavenumber) of G-D beamlets can be used to extract local and directional information of the wavefields and reflections during the migration procedure. This is particularly desirable for target-oriented imaging and various studies on fine-scale structures and angle-domain analyses.

Here we give a detailed description of the G-D beamlet wavefield extrapolation and migration method and apply it to target-oriented imaging. For clarity, we list in Table 1 the major notations. The feasibility of using a G-D beamlet propagator for prestack depth migration is tested through comparative studies of the images produced by beamlet propagators versus global propagators for both the SEG-EAGE salt model 2D prestack data set (Aminzadeh et al., 1994; Aminzadeh et al., 1995; SEG/EAGE 3-D Modeling Committee, 1994) and the Marmousi model data set (Versteeg and Grau, 1991). As a numeric example of target-oriented imaging, we construct local-angle image matrices and the corresponding common-dip-angle image (CDAI) gathers for the 2D SEG-EAGE salt model. We perform structure-based partial summation of the CDAI gathers to improve the image quality of the targets.

### Gabor-Daubechies Beamlet Wavefield Extrapolation

For demonstration purposes, we limit our derivation to the 2D $(x, z)$ case. Generalization to the 3D case is straightforward but involves more technical considerations for practical implementation. In the frequency-space domain, the scalar wave equation can be written as

$$
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v^2(x, z)} \right] u(x, z, \omega) = 0,
$$

where $\omega$ denotes the circular frequency; $v(x, z)$ is the space-domain velocity field, and $u(x, z, \omega)$ represents the pressure field in the frequency-space domain.

With the G-D frame vectors, the wavefield at depth $z$ can be decomposed into beamlets (frame vectors) with windows along the $x$-axis (see Appendix A):

$$
u(x, z, \omega) = \sum_m \sum_n \langle u, \tilde{g}_{mn} \rangle g_{mn}(x)$$

$$= \sum_m \sum_n u_{mn}(z, \omega) g_{mn}(x),
$$

where $u_{mn}(z, \omega)$ is the beamlet decomposition coefficient. The corresponding frame vector $g_{mn}(x)$,

$$g_{mn}(x) = g(x - n \Delta_x, \omega) e^{im \Delta_x},
$$

is a Gaussian-windowed harmonic function with space locus of $\bar{x}_n = n \Delta_x$ and wavenumber locus of $\bar{\xi}_m = m \Delta_x$. Here, $\Delta_x$ and $\Delta_w$ stand for the space and wavenumber sampling intervals of the frame vectors, respectively; $\tilde{g}_{mn}$ is the dual frame vector of $g_{mn}$, constructed in a same way as $g_{mn}$ but with a dual-window function $\tilde{g}(x)$ instead of the Gaussian window $g(x)$ in equation 3. Dual frame vectors are required for redundant G-D frame decomposition and reconstruction. Detailed descriptions of the G-D frame parameters, selection of the
We can further decompose \( G^{mn} \) at \( z + \Delta z \) into beamlets to get the wavefield of the Green’s function in the pure beamlet domain:

\[
G^{mn}(x, z, \Delta z, \omega) = \sum_{j} \sum_{l} (G^{mn}, \tilde{g}_{jl})g_{jl}(x)
\]

\[
= \sum_{j} \sum_{l} P_{jl}^{mn}(z, \Delta z, \omega) g_{jl}(x),
\]  

where \( P_{jl}^{mn}(z, \Delta z, \omega) \) stands for the Green’s function (beamlet propagator) in the beamlet domain. Substituting equation 5 into (4), we get

\[
u(x, z + \Delta z, \omega) = \sum_{m} \sum_{n} u_{mn}(z, \omega) G^{mn}(x, z, \Delta z, \omega).
\]

### Table 1. Notations used in G-D frame representation and derivation for G-D beamlet wavefield extrapolation and migration.

<table>
<thead>
<tr>
<th>Notations</th>
<th>Description</th>
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<td>( \Delta_1 )</td>
<td>Space sampling interval of frame vectors</td>
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<tr>
<td>( \Delta_4 )</td>
<td>Wavenumber sampling interval of frame vectors</td>
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<td>( g(x), \tilde{g}(\xi) )</td>
<td>Gaussian window function and its Fourier transform</td>
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<td>( \tilde{g}(x), \tilde{g}(\xi) )</td>
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<tr>
<td>( g_{mn}, P_{mn} )</td>
<td>Beamlet-domain free propagator</td>
</tr>
<tr>
<td>( \theta_m, \xi_m, v, k(x, z) )</td>
<td>Space-domain perturbation operator</td>
</tr>
<tr>
<td>( \tilde{v}(\tilde{x}_n, z) )</td>
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<tr>
<td>( \tilde{k}_0(\tilde{x}_n, z) )</td>
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<tr>
<td>( \tilde{\xi}_n, \xi_n, \tilde{v}(\tilde{x}_n, z) )</td>
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</tr>
<tr>
<td>( \tilde{\theta}_m, \theta_m, \theta_j, \theta_n )</td>
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<td>( \tilde{\theta}_m, \theta_m, \theta_j, \theta_n )</td>
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<td>( \tilde{\theta}_m, \theta_m, \theta_j, \theta_n )</td>
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<td>( \tilde{\theta}_m, \theta_m, \theta_j, \theta_n )</td>
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<td>( \tilde{\theta}_m, \theta_m, \theta_j, \theta_n )</td>
<td>Image of total strength from prestack depth migration</td>
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into the right-hand side of equation 4 yields

\[ u(x, z + \Delta z, \omega) = \sum_m \sum_n u_{mn}(z, \omega) \sum_l \sum_j P_{mn}^{jl}(z, \Delta z, \omega) g_{jl}(x) \]

\[ = \sum_j \sum_l \left[ \sum_m \sum_n P_{mn}^{jl}(z, \Delta z, \omega) u_{mn}(z, \omega) \right] g_{jl}(x). \]  

(6)

Since the G-D frame representation of the wavefield \( u(x, z + \Delta z, \omega) \) can be expressed as

\[ u(x, z + \Delta z, \omega) = \sum_j \sum_l u_{jl}(z + \Delta z, \omega) g_{jl}(x), \]

we have

\[ u_{jl}(z + \Delta z_j, \omega) = \sum_m \sum_n P_{mn}^{jl}(z, \Delta z, \omega) u_{mn}(\omega). \]  

(8)

Therefore, \( P_{mn}^{jl}(z, \Delta z, \omega) \) will serve as the wavefield propagator in the beamlet domain, which enables calculation of both individual beamlet propagation and the crosscoupling in propagation between different beamlets.

From equations 6 and 8, we see that the accuracy of the reconstructed wavefield depends on the accuracy of the beamlet propagator \( P_{mn}^{jl}(z, \Delta z, \omega) \). For large lateral velocity contrasts, the localized beamlet decomposition of wavefields allows us to deal with the lateral velocity variations locally when constructing the beamlet propagator. First, we introduce a background velocity for each beamlet window. For example, in the \( n \)th window the background velocity \( \bar{v}(\bar{x}_n, z) \) is selected usually as the average or minimum velocity within the window. In this way, local-perturbations can be calculated from the following relation:

\[ k^2(x, z) = k_0^2(\bar{x}_n, z) + [k^2(x, z) - k_0^2(\bar{x}_n, z)], \]  

(9)

where \( k(x, z) = \omega / |v(x, z)| \) is the actual wavenumber and \( k_0(\bar{x}_n, z) = \omega / |\bar{v}(\bar{x}_n, z)| \) is the local background (reference) wavenumber. The second term on the right-hand side of equation 9 gives the corresponding local perturbations. Substituting field-decomposition equation 2 and local-perturbation equation 9 into wave-equation 1 yields the beamlet-domain wave equation,

\[ \sum_n \left( \partial_z^2 + A_n^2 \right) \sum_m u_{mn}(z, \omega) g_{mn}(x) = 0, \]  

(10)

where \( A_n \) is the square root operator:

\[ A_n = \sqrt{\partial_z^2 + k_0^2(\bar{x}_n, z) + [k^2(x, z) - k_0^2(\bar{x}_n, z)]}. \]  

(11)

Then we invoke the one-way wave approximation, which neglects interactions between the forward-scattered and backscattered waves, and derive a formal solution for the evolution of beamlets (windowed wavefields):

\[ G_{mn}^{mn}(x, z, \Delta z, \omega) = e^{\pm iA_n \Delta z} g_{mn}(x). \]  

(12)

Here, \( e^{\pm iA_n \Delta z} \) represents a thin-slab propagator (Wu and de Hoop, 1996; de Hoop et al., 2000) for the beamlets under an assumption of vertical homogeneity within the thin slab of \( \Delta z \) in thickness. Such a vertical-homogeneity assumption is commonly adopted in various recursive depth-wavefield extrapolation algorithms, such as those primarily implemented in the frequency-space domain (Holberg, 1988; Blaquière et al., 1989). The sign of the superscript corresponds to upward (−) or downward (+) propagation. For simplicity, we drop the superscript hereafter. The square root operator \( A_n \), which is actually a pseudodifferential operator, can be approximated in the same manner as in the expansion of GSPs (Xie and Wu, 1998; de Hoop et al., 2000; Jin et al., 2002; Wu, 2003).

We adopt a local perturbation scheme in which the reference velocity varies with the window, instead of using a global background velocity as in GSP methods. Since beamlets have significant values only within neighboring windows, and their spatial spreads are small for short propagation distances, the term \( k^2(x, z) - k_0^2(\bar{x}_n, z) \) in equations 9 and 11 can be treated as a small perturbation. Then, the first-order expansion with a small-angle approximation is sufficient to ensure high accuracy and can yield an approximate expression for \( A_n \). As a result, \( A_n \) can be written as a sum of background and perturbation parts (see Appendix B):

\[ A_n(x, z) \approx \sqrt{\partial_z^2 + k_0^2(\bar{x}_n, z) + [k(x, z) - k_0(\bar{x}_n, z)]} = \sqrt{\partial_z^2 + k_0^2(\bar{x}_n, z)} + \Delta k_n(x, z). \]  

(13)

Accordingly, the thin-slab propagator \( e^{\pm iA_n \Delta z} \) is separated into two parts: a wavenumber-domain-free propagator \( e^{\pm i(\xi, z) \Delta z} \), accounting for one-way wave propagation through the local homogeneous medium, and a space-domain perturbation operator \( e^{\pm i\Delta k / \Delta z} \), accounting for the phase correction for local velocity perturbations (Appendix B). For the free propagator \( e^{\pm i(\xi, z) \Delta z} \), \( \xi_n(z) = \sqrt{k_0^2(\bar{x}_n, z) - \xi^2} \) is the local vertical wavenumber corresponding to a transverse wavenumber \( \xi \) and the local background wavenumber \( k_0(\bar{x}_n, z) \).

Following the decomposition of the thin-slab propagator, the Green’s function \( G^{mn} \) in equation 12 can be constructed as

\[ G^{mn}(x, z, \Delta z, \omega) = G_0^{mn}(x, z, \Delta z, \omega) e^{\pm i\Delta k_n(x, z) \Delta z}. \]  

(14)

Applying spatial Fourier transforms to \( G_0^{mn}(x, z, \Delta z, \omega) \) results in

\[ \tilde{G}_0^{mn}(\xi, z, \Delta z, \omega) = e^{i(\xi, z) \Delta z} \tilde{g}_{mn}(\xi), \]  

(15)

where \( \tilde{G}_0^{mn}(\xi, z, \Delta z, \omega) \) and \( \tilde{g}_{mn}(\xi) \) are the spatial Fourier transforms of \( G_0^{mn}(x, z, \Delta z, \omega) \) and \( g_{mn}(x) \), respectively. The corresponding beamlet propagator defined in equation 5 is then decomposed into a free propagator and a perturbation operator. The G-D beamlet-domain free propagator can be expressed analytically (for details, see Appendix C) as

\[ P_{00}^{mn}(z, \Delta z, \omega) = \left( \tilde{G}_0^{mn}(\xi, \tilde{x}_n, \tilde{\omega}) \right) \int d\xi \tilde{g}(\xi - \xi_n) \times \tilde{g}^{*}(\xi - \xi_{\tilde{n}}) e^{i(\xi, \tilde{x}_n, \tilde{\omega}) \Delta z}, \]  

where \( \tilde{g}(\xi) \) and \( \tilde{g}(\xi) \) are the Fourier transforms of the Gaussian window \( g(x) \), and the dual-window \( \tilde{g}(x) \), \( \tilde{x}_n, \tilde{\xi}_n \) are the \( n \)th and \( n \)th window locations (centers), and \( \tilde{\xi}_n, \tilde{\xi}_n \) are the \( n \)th and
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For the perturbation operator, we adopt the first-order approximation (phase-screen correction) in the local space-beamlet mixed domain and derive the phase-corrected beamlet coefficients (Appendix D):

\[ u^p_{mn}(z, \omega) = \int dx u(x, z, \omega) e^{i \Delta k(x, \bar{x}_n, \omega) \Delta z} \times \tilde{g}(x - \bar{x}_n) e^{-i \bar{\xi}_m x}, \quad (17) \]

where the superscript \( p \) denotes phase correction. The wavefield in the space domain is then constructed as

\[ u(x, z + \Delta z, \omega) \approx \sum_j \sum_l \left[ \sum_n \sum_m P^0_{0jl}(z, \Delta z, \omega) \times u^p_{mn}(z, \omega) \right] g_{jl}(x). \quad (18) \]

With the propagator decomposition mentioned above, one can achieve wavefield extrapolation by first applying the perturbation operation in the mixed domain and then performing the wavefield propagation in the beamlet domain with the local background medium. For local perturbations, the perturbation correction is relatively small and the free propagation plays a dominant role in the procedure.

In addition to frequency, the redundancy ratio, which quantitatively measures the redundancy of the G-D frame (see Appendix A), is an important factor influencing the properties of the beamlet-domain free propagator. Figures 1a–d show the G-D beamlet free-propagator matrices in a homogeneous medium for different frequencies (\( f = 5, 25 \) Hz) and different redundancy ratios (\( R = 2, 4 \)). In the presentation, the elements in a 4D propagator matrix \( \{P^0_{0jl}: 0 \leq j < N_k, 0 \leq l \leq N_w, 0 \leq m < N_k, 0 \leq n \leq N_w\} \) are sorted into a 2D matrix \( P_0(r, c) \), of which the row index \( r = (l - 1) \times N_k + j \) and the column index \( c = (n - 1) \times N_k + m \) represent the beamlet indices for input and output beamlets of the one-step propagation, respectively. Only those elements that are larger than the specific threshold (0.1% of the largest element) are shown and considered as effective elements in the implementation of beamlet wavefield extrapolation. The ratios of effective element number to total element number of the propagator matrices are about 7%, 10%, 5%, and 6% for (a), (b), (c), and (d), respectively.

We see that the free propagators are all highly sparse matrices, the sparsity becoming higher as the redundancy ratio increases but decreasing with frequency. Although higher redundancy ratios result in sparser propagator matrices and lead to better localizations of the wavefield (Appendix A), the rapidly increased element number of the propagator matrix, which is proportional to the square of the beamlet coefficient number and thus to the square of the redundancy ratio — and the associated computation costs — prohibit efficient
implementation of beamlet propagation. From our numeric experiments, a redundancy ratio of two has moderate computation expense yet maintains good localization of the wavefield. Therefore, $R$ is always set to a value of two in the following numerical tests.

The high concentration of the elements within the diagonal band of the free beamlet propagators in Figures 1a–d indicates the propagation coupling of beamlets is significant only between neighboring windows. To investigate in detail the beamlet coupling between different wavenumbers, we pick two window blocks: One on the diagonal and the other off diagonal from the propagator matrices of $R = 2$ (marked as rectangles in Figures 1a, b). The element distributions of the two blocks are shown in Figures 1e–h. Figures 1e and 1f correspond to the self- and cross-coupling of beamlets within one window (diagonal block), and Figures 1g and 1h illustrate the cross-coupling of beamlets of two windows with different space locations (off-diagonal block). For all cases, coupling occurs only between beamlets of adjacent wavenumbers as manifested by the concentration of elements along the diagonal or at the corner of the minor diagonal.

We summarize our G-D beamlet wavefield-extrapolation scheme as follows. At the beginning of the procedure, the parameters of the G-D frame — such as the sampling intervals $\Delta_x$ and $\Delta_\xi$, the numbers of windows and local wavenumbers in each window $N_w$ and $N_k$ etc., and the length and shape of the Gaussian window $g(x)$ and its dual-window $\tilde{g}(x)$ — are determined based on the frame-construction principle and the procedure described in Appendix A as well as according to the structural properties of the model considered. At each depth, a local phase-screen correction is applied to the space-domain wavefield $u(x, z, \omega)$, and the corrected wavefield is decomposed into beamlets based on equation 17 to produce the modified beamlet coefficients $\hat{u}_{mn}^p$. Then the beamlet free propagator $P_{mn}^0$ is calculated from equation 16. Finally, the wavefield at the next depth is obtained by substituting $\hat{u}_{mn}^p$ and $P_{mn}^0$ into equation 18. Such a three-step procedure is carried out iteratively to construct the wavefield for the whole space.

**PROPAGATION OF SINGLE G-D BEAMLET SOURCES AT THE SURFACE**

Using the beamlet-domain wavefield-extrapolation algorithm, through forward modeling we investigate the propagation features of individual G-D beamlet sources excited on the surface. Three velocity models are considered here: a homogeneous model, a layered model with one layer of laterally varying velocities sandwiched by two homogeneous layers (Figure 2a), and the complex SEG-EAGE 2D salt model containing a high-velocity salt body with complicated top surface (Figure 2b). The wavefields of single beamlet sources with different frequencies are shown in Figures 3–5 for the three models. With the well-constructed beamlet propagator, the wavefields are propagated accurately, even for
the highly heterogeneous case (Figure 5). The beamlet-source fields clearly show space expansion along the propagating directions and exhibit severe wavefront distortion in accordance with the velocity jump at sharp boundaries. The higher the frequency, the narrower the spread of the wavefield. The wavefront distortion, in contrast, appears to be of comparable degree at different frequencies (Figures 4 and 5).

In some beam-based wavefield extrapolation techniques, such as the Gaussian beam method (Červený et al., 1982; Nowack and Aki, 1984; Hill, 1990, 2001), the wavefield is decomposed into beams only at the initial depth (the surface), and the beams are extrapolated directly to the model space without iterative wavefield decomposition and reconstruction. For a large extrapolation distance, the initially localized beams gradually lose their localizations. For the Gaussian beam method, which uses the parabolic approximation and the ray approximation in wavefield extrapolation, the beam distortion and abnormal expansion would become more severe along propagating paths. In the beamlet method, the space-domain wavefield is decomposed into beamlets, propagated in the beamlet domain (plus a phase-correction in the local space-beamlet mixed domain), and reconstructed from beamlets at every depth. In this respect, the beamlet method should be more accurate for wavefield propagation. More importantly, the method allows a better means of extracting local directional information of the wavefield at each extrapolation step. This feature is particularly advantageous for directivity-involved analysis and angle-domain local structural imaging.

**PRESTACK G-D BEAMLET MIGRATION FOR THE 2D SEG-EAGE SALT MODEL AND THE MARMOUSI MODEL**

Poststack G-D beamlet depth migration has been conducted by Wu and Chen (2001, 2002a) to show the validity and feasibility of G-D beamlet propagators applied to seismic imaging. In this paper, we perform common-shot prestack beamlet migration for the 2D SEG-EAGE salt model and the Marmousi model. While the former model is characterized by a strong velocity contrast and large dips of subsalt faults, the latter model is structurally complex, with many thin layers broken by several major faults and an unconformity surface.

Considering the high velocity contrast between the salt body and the surroundings, we choose narrower windows in the beamlet decomposition of wavefields for the SEG-EAGE salt model than those for the Marmousi model to reduce lateral perturbations in each window. The parameters of the G-D beamlets used as well as the information for both of the models and the prestack data sets are listed in Table 2. In common-shot migration, the source field $u^S(x, z = 0, \omega)$ for each shot is forward-propagated, and the corresponding recorded data field $u^R(x, z = 0, \omega)$ is back-propagated to the image space in accordance with equation 18. Here, $S$ is the shot index and $R_s$ stands for the receiver array for shot $S$. At each depth, the traditional space-domain image condition (Claerbout, 1971)
is used to construct the single-shot image, and the final image is obtained by superimposing all of the single-shot images together:

\[ I(x, z) = \sum_S I^S(x, z) \]

\[ = \sum_S \text{Re} \left( \sum_\omega u^S(x, z, \omega) u^R_\omega(x, z, \omega) \right) \], \hspace{1cm} (19)

where \( I \) is the final image and \( I^S \) is the single-shot image from shot \( S \). The term \( \text{Re} \) stands for the real part of the complex field.

Figure 6a shows the image for the SEG-EAGE salt model obtained using the G-D beamlet propagator. For comparison, we also plot the images migrated using the GSP (Xie et al., 2000) in Figure 6b. All of the main features, including the boundaries of the salt body, sharp edges, steep faults above the salt body, and even parts of the subsalt steep reflectors, are imaged clearly with both the beamlet method and the screen method. The overall image quality of the two methods is comparable except in the interior of the salt body, where the GSP migrated image has a slightly higher noise level.

The intricate structure of the Marmousi model (Figure 7a) produces complicated seismic data, making it difficult to generate images of high quality by many prestack migration methods (Gray et al., 2001). The prestack data set for this model continues to be used as a test bed for migration and velocity-estimation methods. Marmousi images have been obtained using various migration schemes (Geoltrain and Brac, 1993; Gray and May, 1994; Nichols, 1996; Audebert et al., 1997; Han, 1998; Operto et al., 1998; Huang et al., 1999; Hill, 2001; Kühl et al., 2001). We perform prestack G-D beamlet migration for the model and obtain the image shown in Figure 7b. The image from the GSP (Xiaobi Xie, 2002, personal communication) is also plotted in Figure 7c for comparison. Major faults and many thin layers are well imaged by both methods. Again, a lower noise level is achieved with beamlet migration, especially at the upper and middle parts of the image.

Since G-D frame decomposition of wavefields is overcomplete and nonorthogonal, more computations and storage space are required for G-D beamlet-based wavefield propagation and migration. As a result, computation of beamlet migration is several times slower than that of the GSP method. Image comparisons for the two models, however, indicate that prestack migration can produce images of comparable — even higher — quality for complicated structures. More importantly, the localizations of the wavefield decomposition and propagation in the G-D beamlet domain allow detailed study of the directivity-involved features of local structures. This is particularly desirable for target-oriented prestack migration and imaging.

**TARGET-ORIENTED PRESTACK G-D BEAMLET MIGRATION FOR THE 2D SEG-EAGE SALT MODEL**

In consideration of the huge number of computations required in full prestack migration, especially for 3D surveys, Rietveld et al. (1992) and Rietveld and Berkhout (1994) propose

![Figure 6. Prestack migration images for the SEG-EAGE salt model (a) by G-D beamlet propagator and (b) by generalized screen propagator (GSP) (Xie et al., 2000).](image)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model</th>
<th>Data</th>
<th>G-D beamlet</th>
</tr>
</thead>
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<td>( N_x )</td>
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<td>645</td>
<td>176</td>
</tr>
<tr>
<td>( x ) dx (m)</td>
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<td>24.4</td>
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<tr>
<td>( N_k )</td>
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</table>

\( N_x, N_z \) are the sample numbers of the model in \( x \) and \( z \) directions with \( dx \) and \( dz \) as the sampling interval, respectively. For the prestack data, \( N_x \) is the shot number and \( L_{\text{tr}} \) is the trace (receiver) number for each shot; \( N_t \) and \( dt \) are the length and sampling interval in time of the traces. In G-D beamlet decomposition and propagation, the wavefield at each depth is zero-padded from \( x \)-width of \( L_{\text{tr}} \) to \( L_x \). Since the redundancy ratio of G-D frame is kept to be 2, the number of wavefield decomposition coefficients \( N_{\text{coef}} \) is always equal to \( 2L_x \).
an efficient as well as accurate method to migrate prestack data in a target-oriented way by means of controlled illumination. As a special case of the general approach of areal shot-record migration (Berkhout, 1992), controlled illumination migration is superior to conventional plane-wave stacking (Taner, 1976; Schultz and Claerbout, 1978) in the sense that control of the source wavefield is put at the target. Another target-illumination-based prestack migration method is proposed by Wu et al. (2002) using partial sources. In that method, both image quality and computation efficiency are improved through selecting only a portion of the sources that contribute illumination energies to the target structures in prestack migration. The principle and procedure of partial source migration apply to any kind of wavefield extrapolation method. In this study, we propose performing target-oriented migration by means of local-angle-domain imaging and partial superposition of common-dip-angle images based on the target structures. In this way, the dual-domain localization property of G-D beamlet decomposition and propagation can be utilized fully, and image quality improvement can be achieved by controlling the image at the target.

**Local-angle-domain prestack G-D beamlet migration**

In prestack G-D beamlet migration, the incident field (from sources) as well as the scattered field (from receivers) at each extrapolation step (each depth) are obtained as the superposition of contributions from all of the G-D beamlets as expressed in equations 2 or 7. By superimposing only the beamlets with the same local wavenumbers, we get the incident and scattered fields for each individual wavenumber:

\[
\begin{align*}
    u^S(x, z, \xi_m, \omega) &= \sum_n \sum_{\omega} u^S_{mn}(z, \omega) g_{mn}(x) \\
    u^R(x, z, \xi_j, \omega) &= \sum_l \sum_{\omega} u^R_{lj}(z, \omega) g_{lj}(x),
\end{align*}
\]

where \( u^S_{mn}(z, \omega) \) and \( u^R_{lj}(z, \omega) \) are beamlet coefficients of the incident field and scattered field at depth \( z \), respectively. The obtained values \( u^S(x, z, \xi_m, \omega) \) and \( u^R(x, z, \xi_j, \omega) \) are called the directional incident and scattered wavefields at points \( (x, z) \), since a local wavenumber \( \xi_j \) corresponds directly to a propagating angle

\[
\hat{\theta}_j = \sin^{-1} \left[ \frac{\xi_j}{k(x, z)} \right],
\]

where \( \hat{\theta}_j \) is the angle with respect to the vertical. According to equation 21, the directional incident and scattered wavefields can be expressed as \( u^S(x, z, \hat{\theta}_m, \omega) \) and \( u^R(x, z, \hat{\theta}_j, \omega) \) in terms of their propagating angles. Substituting \( u^S(x, z, \hat{\theta}_m, \omega) \) and \( u^R(x, z, \hat{\theta}_j, \omega) \) into the imaging condition 19 in place of the total incident and scattered fields \( u^S(x, z, \omega) \) and \( u^R(x, z, \omega) \), and considering the coordinate transform from \( \xi_m, \xi_j \) to \( \hat{\theta}_m, \hat{\theta}_j \),

\[
\begin{align*}
    \xi_m &= k \sin \hat{\theta}_m, \quad |d\xi_m| = |d\hat{\theta}_m| k \cos \hat{\theta}_m \\
    \xi_j &= k \sin \hat{\theta}_j, \quad |d\xi_j| = |d\hat{\theta}_j| k \cos \hat{\theta}_j,
\end{align*}
\]

with \( k(x, z) = \omega/v(x, z) \) as the wavenumber, the local-angle image matrix is calculated as

\[
I(\hat{\theta}_m, \hat{\theta}_j, x, z) = \cos \hat{\theta}_m \cos \hat{\theta}_j \sum_{\omega} k^2(x, z) x^S(x, z, \hat{\theta}_m, \omega) u^R(x, z, \hat{\theta}_j, \omega). \]

Local-angle-image matrix \( I(\hat{\theta}_m, \hat{\theta}_j, x, z) \) measures the contributions to the final image from different incident field/scattered field pairs. We can output images of different incident

Figure 7. Prestack migration images for the 2D Marmousi model. (a) Velocity model; (b) G-D beamlet migration image; (c) GSP migration image (Xiaobi Xie, 2002, personal communication).
angle/scattering angle pairs as an image album. Figure 8 shows the image album for some angle pairs ($\bar{\theta}_m, \bar{\theta}_j$) by common-shot prestack migration for the SEG-EAGE salt model. Here, positive angles represent propagating directions from vertical to the right, and negative angles represent propagating directions from vertical to the left. It is noteworthy that the individual images in the figure highlight the structural features with different orientations. The true dips of the enhanced structures are close to the favored dips predicted based on the plane-wave reflection theory from the propagating directions of the incident-scattered field pairs (marked alongside each panel).

**Target structure-oriented partial dip migration**

Assuming that the local scatterer is a planar reflector, the local-angle-image matrix (2D matrix) can be simplified to different single-variable functions by various stacking methods. Through the coordinate transform from $(\bar{\theta}_m, \bar{\theta}_j)$ into $(\bar{\theta}_n, \bar{\theta}_r)$,

\[
\bar{\theta}_n = \frac{\bar{\theta}_j + \bar{\theta}_m}{2} \quad \text{and} \quad \bar{\theta}_r = \frac{\bar{\theta}_j - \bar{\theta}_m}{2},
\]

the local-angle-image matrix $I(\bar{\theta}_m, \bar{\theta}_j, x, z)$ is then converted to $I(\bar{\theta}_n, \bar{\theta}_r, x, z)$, with the normal of the reflector $\bar{\theta}_n$ and the reflection angle with respect to the normal $\bar{\theta}_r$, replacing the incident and scattering angles $\bar{\theta}_m$ and $\bar{\theta}_j$. On the one hand, images from different normal angles for a fixed common reflection angle can be stacked together to form a common reflection-angle image (CRAI) gather,

\[
I(\bar{\theta}_r, x, z) = \sum_{\bar{\theta}_n} I(\bar{\theta}_n, \bar{\theta}_r, x, z),
\]

which can be used directly for amplitude variation with angle (AVA) analysis, migration velocity analysis, and estimation of changes in the elastic parameters (Ursin et al., 1996; Xu et al., 1998; Brandsberg-Dahl et al., 1999, 2003). On the other hand, because of the mirror reflection of planar interfaces, we can add all of the contributions from different reflection angles, resulting in a set of images as a function of the normal angle of local interfaces:

\[
I(\bar{\theta}_n, x, z) = \sum_{\bar{\theta}_r} I(\bar{\theta}_n, \bar{\theta}_r, x, z),
\]

This kind of common-angle image gather can be called a common dip-angle image (CDAI) gather since one normal angle corresponds uniquely to a dip angle for a planar interface. The final image of total strength can be obtained by summing up either the CRAI or the CDAI gather:

\[
I(x, z) = \sum_{\bar{\theta}_r} I(\bar{\theta}_r, x, z) = \sum_{\bar{\theta}_n} I(\bar{\theta}_n, x, z).
\]

In this study, for the purpose of target-oriented imaging, we consider only the features and applications of CDAI gatherers. For local planar reflectors, ideally the CDAI gatherers will have peaks at the corresponding real geologic dips. Figure 9 shows the CDAI gatherers in the

**Figure 8.** Local-angle-image album with different $(\bar{\theta}_m, \bar{\theta}_j)$ pairs for the SEG-EAGE salt model obtained by common-shot prestack G-D beamlet migration: (a) $(0^\circ, 0^\circ)$; (b) $(0^\circ, -60^\circ)$; (c) $(0^\circ, 60^\circ)$; (d) $(30^\circ, -30^\circ)$; (e) $(-30^\circ, -60^\circ)$; (f) $(30^\circ, 60^\circ)$.

**Figure 9.** CRAI gathers in the form of rose diagrams for the SEG-EAGE salt model.
form of rose diagrams for the SEG-EAGE salt model. At each point a rose diagram is plotted, with each of the 11 petals representing one dip direction. The length of the petal is proportional to the image strength of the corresponding dip. The total image strength for all dip directions is given by the number near each point.

From the figure we can see that most of the image strength comes from the contributions of dip directions in the vicinity of the real dip angle for most of the structures, such as those outside the salt body and the boundary of the salt body. In the subsalt area, however, especially for the three steep faults, not only are the total image strengths much weaker, but the maximum-strength dip directions also deviate significantly from the real dips at the upper parts of the faults, coinciding with the relatively poor image quality for these structures as shown in Figure 6. The large discrepancy between the imaged dips of maximum strength and the real dips is likely the consequence of the limited acquisition aperture and the overlying high-velocity salt body that prevents a considerable amount of downward-propagated energy from reaching the subsalt area (Wu and Chen, 2002b).

CDAI gathers also can be viewed as an image album with different dips. Figure 10 gives part of the CDAI album for the subsalt structures. In each dip-angle image, structures with specific directivity features are strengthened while others are relatively weakened. The three subsalt steep faults, taken as our target of imaging, are present explicitly in Figures 10b and 10c but can hardly be detected in other panels. To image structures of different directivity features, Brandsberg-Dahl et al. (2003) propose a focusing-in-dip method using cutoff functions in the integration over migration dips to obtain noise-suppressed common scattering-angle image gathers of interesting structures. That technique also provides a tool to estimate local geologic dips and improve image quality.

For complex media such as the subsalt structures of the SEG-EAGE salt model, however, local geologic dips may vary from point to point, which makes it difficult to use the focusing-in-dip procedure to obtain a consistent image for neighboring structures. Considering the real dips of about 32° for the left fault and 44° for the middle and the right faults beneath the salt body, here we superimpose the CDAI with dip angles ranging from 16° to 60° to better image the target faults. In Figure 11, we compare the resultant subsalt image (Figure 11b) with the original migration image (Figure 11a, a zoom of the target area from Figure 6a). Since the contributions of only favorable dip angles are superimposed, the coherent noises, especially those that intersect with the left steep fault, are suppressed markedly. Some other structures, with directivity features much different from those of the steep faults, are also weakened, considerably enhancing the steep faults (Figure 11b). This demonstrates the feasibility of such a structural dip-based CDAI partial summation approach for target-oriented imaging.

In this proposed angle-domain migration and target-oriented imaging scheme, the wavefield extrapolation procedure is similar to that of the common-shot prestack G-D beamlet migration. The differences lie only in two aspects: (1) Partial wavefield reconstruction in the local-angle domain is performed instead of full wavefield reconstruction in the space domain to derive the directional incident and scattered fields, and (2) angle-domain contributions are extracted and partially superimposed in imaging at the target. This method clearly does not reduce migration computation costs. However, source field controls [such as those proposed by Rietveld et al. (1992), Rietveld and Berkhout (1994), and Wu et al. (2002)] and simultaneous imaging controls for the target (i.e., combining the target-illumination-based migration method and the present target-structure-based partial CDAI summation approach) might provide possible ways to improve image quality with high computation efficiency in target-oriented prestack depth migration.

Figure 10. Common-dip-angle image (CDAI) album for the subsalt structure. (a) Horizontal dip; (b) +30°; (c) +45°; (d) −30°; (e) −45°; (f) total image.
Figure 11. Comparison of the subsalt image obtained by (b) structure-based partial CDAI superposition (16°–60°) with the original image (a) by G-D beamlet prestack migration.

CONCLUSIONS

We have studied and tested beamlet propagation and imaging using Gabor-Daubechies frame propagators for synthetic data sets. The use of local background velocities and local perturbations allows for optimization of the local beamlet propagators and easy handling of the strong lateral velocity variations. The analytical forms of the G-D frame propagator and phase-correction operator are derived based on the one-way operator decomposition and screen approximation. The prestack data sets of the 2D SEG-EAGE salt model and the Marmousi model are tested by using G-D beamlet propagators in prestack depth migration. The resultant high-quality images demonstrate the capability and potential of the method for imaging complex structures.

Owing to the localization property in both space and direction, beamlet decomposition and extrapolation of wavefields provide localized directional information in each migration step from which local-angle image matrices can be constructed to quantify contributions from different incident-scattered field pairs to the final image. The directional features can also be represented by CDAI gathers through a coordinate transformation and partial summation of the local-angle image matrices. Image quality for the target can be improved by controlled superposition of CDAI based on the directivity feature of target structures. Numerical tests on the 2D SEG-EAGE salt model prestack data show the feasibility and potential of G-D beamlet migration in target-oriented imaging.

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APPENDIX A

GABOR-DAUBECHIES FRAME REPRESENTATION

Since the windowed Fourier transform (WFT) is highly redundant in signal decomposition, and the subsequent WFT reconstruction is very time consuming, many studies have focused on developing sparsely sampled yet accurate decomposition/reconstruction schemes for various signals. The windowed Fourier frame representation, a discretely sampled but complete and stable version of the WFT, was thus developed to overcome the difficulties encountered with the WFT (Daubechies, 1992; Mallat, 1998). In such a representation, an arbitrary function \( f(x) \) can be expanded by a set of windowed Fourier frame functions (vectors) \( \{ g_{mn}(x) \} \) that are constructed by space (time) shifting and harmonically modulating a window function \( g(x) \):

\[
\hat{f}(x) = \sum_m \sum_n \hat{f}_{mn} g_{mn}(x) = \sum_m \sum_n \hat{f}_{mn} g(x - n\Delta_x)e^{im\Delta_kx}, \quad (A-1)
\]

where \( \Delta_x \) and \( \Delta_k \) are the space (time) and wavenumber (frequency) sampling intervals of the frame vectors and where \( n\Delta_x \) and \( m\Delta_k \) are the corresponding dual-domain loci of the vector \( g_{mn}(x) \). Using the Gaussian window function in equation A-1, Gabor (1946) originally represented signals with sampling intervals satisfying \( \Delta_x\Delta_k = 2\pi \), which was later proven to be the most compact representation of this kind (Daubechies, 1990, 1992). Although the completeness of \( \{ g_{mn}(x) \} \) will be guaranteed if the condition \( \Delta_x\Delta_k \leq 2\pi \) is satisfied, Daubechies derived through the frame formalism that the reconstruction under Gabor’s critical sampling is unstable and that oversampling \( \Delta_x\Delta_k < 2\pi \) must hold for stabilizing the windowed Fourier frame representation (Daubechies, 1990, 1992). Optimally localized in both space (time) and wavenumber (frequency) domains under the Heisenberg uncertainty principle, the Gaussian window is the most favorable for windowed Fourier analysis of signals (Mallat, 1998). The windowed Fourier frame with a Gaussian window was named by Daubechies as the Weyl-Heisenberg coherent state frame. Some authors also call it the Gabor frame (Feichtinger and Strohmer, 1998). Here, we call it the Gabor-Daubechies (G-D) frame to emphasize the contributions that Gabor and Daubechies have made to establishing the theory.

In equation A-1 for the windowed Fourier frame representation, \( \hat{f}_{mn} \) are called the frame decomposition coefficients of the function \( f(x) \). Based upon the frame theorem (Daubechies, 1992; Qian and Chen, 1996; Mallat, 1998), there
exists a dual-window function \( \hat{g}(x) \) so that \( \hat{f}_{mn} \) can be computed by the regular inner product operation, i.e.,

\[
\hat{f}_{mn} = (f, \hat{g}_{mn}) = \int dx f(x) \hat{g}_{mn}^*(x) = \int dx f(x) \hat{g}_{mn}^*(x - n \Delta x) e^{-im \Delta k x}, \quad (A-2)
\]

where \( * \) stands for complex conjugate and \( \hat{g}_{mn}(x) = \hat{g}(x - n \Delta x) e^{im \Delta k x} \) is the dual vector to \( g_{mn}(x) \) in the sense that the double summation results in a delta function:

\[
\sum_m \sum_n g_{mn}(x) \hat{g}_{mn}(x') = \delta(x - x'). \quad (A-3)
\]

The set of dual vectors \( \{ \hat{g}_{mn}(x) \} \) also constructs a windowed Fourier frame (Daubechies, 1992; Mallat, 1998) that is determined uniquely by the dual-window function \( \hat{g}(x) \).

Compared with the critical-sampling case \( (\Delta_s, \Delta_k = 2\pi) \), frame representation with \( \Delta_s, \Delta_k < 2\pi \) is redundant and thus nonorthogonal. The redundancy is measured by the redundancy ratio

\[
R = \frac{2\pi}{\Delta_s \Delta_k}, \quad (A-4)
\]

which is always above the critical value of one.

Redundant frame representations result in a nonunique selection of the dual-window function \( \hat{g}(x) \) and, consequently, nonunique dual frames. For such cases, an optimum dual frame should be determined to meet the requirement of the specific purpose. In this work, we select the dual-window function whose shape is the closest to the Gaussian window in the sense of the least-square error to ensure that the dual frame vectors possess good localizations similar to the original Gaussian-windowed frame vectors. The so-defined optimum dual-window function is constructed based on the method proposed by Qian and Chen (1996).

For comparison, the Gaussian window and its dual-window with different redundancy ratios are plotted in Figure A-1, from which we see that the higher the redundancy ratio, the closer the dual-window function is to the Gaussian window function. From a wave propagation point of view, high redundancy in the G-D frame representation of wavefields leads to good localization in both space (time) and wavenumber (frequency). This is a desirable feature for efficient extrapolation of wavefields. However, the computation expense involved in the frame representation and the associated wavefield extrapolation will rise rapidly with increasing redundancy ratio because of the proportionally increased number of frame vectors and decomposition coefficients. This is shown in Table A-1, where the parameters for G-D frame representation of 1D space-domain signals are listed and compared with those for both the WFT and orthogonal representations. As a result, special considerations are required to provide the possibility of a trade-off between the localization property of wavefields and the computation efficiency. This issue is further addressed in the implementation of beamlet-domain wavefield extrapolation in the main text.

In summary, some desirable properties of the redundant G-D frame representation compared with orthogonal representations, especially for wavefield-related studies are (1) optimal space (time)-wavenumber (frequency) localization;
Asymptotic phase matching in the forward direction, the equation 11 will be

\[ A_n \approx \sqrt{\frac{\partial^2}{\partial x^2} + k_0^2(\tilde{x}_n, z)} + \frac{1}{2} \left[ k^2(x, z) - k_0^2(\tilde{x}_n, z) \right] \]

With the typical small-angle approximation \( \tilde{\alpha}_i^2 \ll k_0^2 \) and asymptotic phase matching in the forward direction, the second term on the right-hand side can be modified to

\[ \approx k(x, z) - k_0(\tilde{x}_n, z) = \Delta k_n(x, z). \] (B-2)

which is the well-known phase-screen correction. Note that the phase-screen correction matches exactly the travel time difference in the forward direction. Combining equations B-1 and B-2 results in equation 13 in the main text.

The thin-slab propagator \( e^{iA_n \Delta z} \) is accordingly split into two parts: (1) a free propagator \( e^{\sqrt{\frac{\partial^2}{\partial x^2} + k_0^2(\tilde{x}_n, z)} \Delta z} \), accounting for wavefield propagation in the local homogeneous medium with \( k_0(\tilde{x}_n, z) \) as the local background wavenumber and (2) a perturbation operator \( e^{i\Delta k_n(x, z) \Delta z} \) for the phase-screen correction. After a Fourier transform with respect to \( x \), the free propagator takes the form of \( e^{\sqrt{k^2(x, z) - k_0^2(\tilde{x}_n, z)} \Delta z} (\xi \) stands for the transverse wavenumber parameter) in the wavenumber domain, in which it can be implemented more efficiently than in the space domain.

**APPENDIX C**

**GABOR-DAUBECHIES BEAMLET-DOMAIN FREE PROPAGATOR**

Applying the inverse Fourier transform to equation 15 yields

\[ G_{0mn}(x, \Delta z, \omega) = \frac{1}{2\pi} \int d\xi e^{i\xi x} e^{i\Delta k_n(\xi)} \hat{g}_{mn}(\xi) \] (C-1)

By definition, the G-D beamlet free propagator \( P_{0jl}^{mn} \) then becomes

\[ P_{0jl}^{mn}(\xi, \Delta z, \omega) = \langle G_{0mn}^j, \hat{g}_{jl}^* \rangle \]

\[ = \frac{1}{2\pi} \int dx \hat{g}_{jl}^*(x) \int d\xi e^{i\xi x} e^{i\Delta k_n(\xi)} \hat{g}_{mn}(\xi) \]

\[ = \frac{1}{2\pi} \int d\xi e^{i\Delta k_n(\xi)} \hat{g}_{mn}(\xi) \hat{g}_{jl}^*(\xi) \]

\[ = \frac{1}{2\pi} \int dx g_{jl}^*(x) e^{i\Delta k_n(\xi)} \hat{g}_{mn}(\xi) \]

\[ \approx \hat{g}_{jl}^*(\xi - \xi_n) e^{i\Delta k_n(\xi)} \hat{g}_{mn}(\xi). \] (C-2)

where

\[ \hat{g}_{jl}^*(\xi - \xi_n) = \int d\xi g_{jl}(x) e^{i\Delta k_n(\xi)} \]

\[ = \int d\xi g_{jl}(x - \xi_n) e^{i\Delta k_n(\xi)} \]

\[ = \hat{g}_{jl}^*(\xi - \xi_n) e^{i\Delta k_n(\xi)} \]

Substituting equations C-3 and C-4 into the last line of equation C-2, we obtain the final expression for the free propagator (equation 16 in the main text).

**APPENDIX D**

**LATERAL-PERTURBATION PHASE CORRECTION**

From equations 4, 14, and 16 and following the derivation of equation 6 in the main text, we have

\[ u(x, z + \Delta z, \omega) = \sum_m \sum_n u_{mn}(z, \omega) \]

\[ \times e^{i\Delta k(\tilde{x}_n, \omega) \Delta z} G_{0mn}^j(x, \Delta z, \omega) \]

\[ = \sum_m \sum_n u_{mn}(z, \omega) e^{i\Delta k(\tilde{x}_n, \omega) \Delta z} \]

\[ \times \sum_l \sum_j P_{0jl}^{mn}(\Delta z, \omega) \hat{g}_{jl}(x) \]

\[ = \sum_j \sum_l \left[ \sum_m \sum_n P_{0jl}^{mn}(\Delta z, \omega) \hat{g}_{jl}(x) \right] \]

In contrast to the free propagation of wavefields, which is implemented in the beamlet domain, the perturbation-related phase correction is carried out in the local space-beamlet mixed domain. The term \( u_{mn}(z, \omega) e^{i\Delta k(\tilde{x}_n, \omega) \Delta z} \) in equation D-1 can be approximated by connecting the lateral perturbations.
within individual windows:

\[ u_{mn}(z, \omega)e^{i\Delta k(z, \tilde{x}_n, \omega)}dx \approx \int dx' u(x', z, \omega)e^{i\Delta k(z, \tilde{x}_n, \omega)}dx' = u^p_{mn}(z, \omega). \]  

The phase-corrected beamlet coefficients \( u^p_{mn} \) are used to account for the effect of lateral heterogeneity and are then applied to beamlet-domain free propagation, as expressed in equation 18 in the main text.

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