Scattering of elastic waves by an elastic or viscoelastic cylinder

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SUMMARY

The scattering of elastic waves by an elastic or viscoelastic cylinder is investigated numerically. The analytical solutions of the scattered and internal fields excited by a normally incident plane P wave or SV wave are derived in a concise form. Solutions for cylindrical P-wave incidence and for some special cases of the cylinder medium and the matrix medium are also evaluated explicitly. Numerical results for directivity patterns, scattering cross-sections and synthetic seismograms are given for both non-absorbing and absorbing cylinders. Results show that the directivity patterns that undergo a mode conversion (the P–SV scattering and the SV–P scattering) are invariant as ka increases, k being the wavenumber and a the cylinder radius, the P–P scattering and the SV–SV scattering concentrate to the forward direction and all scattering waves (P–P, SV–SV and P–SV or SV–P) grow into more complicated structures. The interference of diffracted and transmitted waves causes the maxima and minima in the curves of scattering cross-sections, and the small high-frequency peaks that appear in a low-velocity non-absorbing cylinder correspond to the resonance scattering. The total scattered field is mainly the superposition of the geometrically transmitted waves (P1,P2,P1, P1P2S1 and P1S2S1 for P-wave incidence and S1S2S1, S1S2P1 and S1P2P1 for SV-wave incidence) that go through the cylinder and the diffracted waves (P1,P1,P1 and P1P1S1 for P-wave incidence and S1S1S1 and S1S1P1 for SV-wave incidence) that propagate on the matrix side of the cylinder interface. The arrival times and the waveforms of these different wave pulses depend on the positions of the observation points and the elastic parameters of the medium.

Key words: diffraction, elastic wave theory, scattering, seismograms.

1 INTRODUCTION

The scattering of an elastic wave normally impeding upon an infinitely long elastic cylinder embedded in an elastic medium has been studied for many decades. Exact steady-state solutions have been derived, especially for incident plane elastic waves (White 1958; Pao & Mow 1973; Miklowitz 1978; Bogan & Hinders 1994).

Partly because the analytic solution is lengthy, many of the details of the scattering process have not been studied and clarified. For example, the resonant responses of the cylinder are not known, and the wavefield in the interior of the cylinder, including possible caustics, is not clear. In comparison with the simpler but similar case of scattering of an acoustic wave by the same infinitely long solid cylinder immersed in a liquid (Flax et al. 1981; Numrich & Überall 1992; Überall 1992; Hackman 1993), the present problem of scattering of an elastic wave has been studied far less.

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in cross-section and infinite in length, in perfect contact with the extended solid matrix. The cylinder is generally assumed to be elastic, but to study the effect of the attenuation of the scattering object, the case of a viscoelastic cylinder is also computed. Analytic steady-state solutions, in the form of wave function expansions, are well known, but we attempt to provide a more concise form by eliminating redundant terms. Transient solutions, necessary for the calculation of synthetic seismograms, are obtained by applying a Fourier transform to the steady-state responses, given an input elastic pulse.

In the following, we first derive the analytical expressions of solid-cylinder scattering by planar or cylindrical P-wave incidence. The numerical results are then given for the scattering diagrams, the scattering cross-sections and the seismograms. Since the analyses of SV-wave incidence are similar to those of P-wave incidence, the numerical results for plane SV-wave incidence will be given briefly in Section 3.5.

2 ANALYTIC SOLUTIONS

2.1 Scattered and internal waves

The plane elastic P wave travels in the direction of positive y in a solid elastic medium ‘1’ of infinite extent (Fig. 1) with Lamé constants $\lambda_1$ and $\mu_1$ and density $\rho_1$. The cylinder medium is labelled ‘2’ and has Lamé constants $\lambda_2$ and $\mu_2$ and density $\rho_2$. The displacement potential of the incident P wave with unit amplitude can be expanded in the frequency domain as

$$\Phi_{11}^i = e^{i(k_1 r - \omega t)} = \sum_{m=0}^{\infty} \epsilon_m^i m J_m(k_1 t) e^{-i\omega t}. \tag{1}$$

The resulting scattered field outside and transmitted field inside the cylinder in the frequency domain can be shown from the wave equation to be (Pao & Mow 1973) as follows. The scattered P wave is

$$\Phi_{11}^s = \Phi_1 = \sum_{m=0}^{\infty} A_m H_{11}^m(k_1 t) \cos(\omega t) e^{-i\omega t}, \tag{2a}$$

the scattered SV wave is

$$\psi_{11}^s = \Psi_1 = \sum_{m=1}^{\infty} B_m H_{11}^m(k_1 t) \sin(\omega t) e^{-i\omega t}, \tag{2b}$$

the internal P wave is

$$\Phi_{21}^i = \sum_{m=1}^{\infty} C_m J_m(k_2 t) \cos(\omega t) e^{-i\omega t}, \tag{3a}$$

and the internal SV wave is

$$\psi_{21}^i = \sum_{m=1}^{\infty} D_m J_m(k_2 t) \sin(\omega t) e^{-i\omega t}. \tag{3b}$$

In these equations $(r, \theta, z)$ are the cylindrical coordinates [Fig. 1, the z-axis coinciding with the cylinder axis, $(r, \theta)$ being the polar coordinates in the plane normal to the z-axis and $\theta$ being measured from the positive y-axis], $t$ is the time, $\omega$ is the angular frequency, $J_m$ and $H_{11}^m$ are, respectively, the Bessel function of the first kind and the Hankel function of the first kind, both of order $m$, $\epsilon_m$ is the Neumann factor, $\epsilon_0 = 1$, $\epsilon_m = 2$ for $(m=1, 2, 3, \ldots)$, the $k$ are the wavenumbers, $k_1 = \omega / v_1$, $k_2 = \omega / v_2$, $v_2 = v_1$, with the $v$ being the phase velocities, $v_1$ for the compressional wave and $v_2$ for the shear wave.

Figure 1. Geometry used for formulating the problem of elastic scattering from an infinite circular cylinder.

are expressed in terms of displacement potentials $\Phi$ for the compressional wave and $\Psi$ for the shear wave, which are related to the displacement and stress components in the frequency domain by the expressions

$$U_{it} = \frac{\partial \Phi}{\partial r} + \frac{\partial \Psi}{\partial \theta}, \tag{4a}$$

$$U_{it} = \frac{\partial \Phi}{\partial r} + \frac{\partial \Psi}{\partial \theta}, \tag{4b}$$

$$\sigma_{rel} = \lambda_1 \nabla^2 \Phi_1 + 2 \mu_1 \left( \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) + \mu_1 \left( \frac{\partial^2 \Psi}{\partial r^2} - \mu_1 \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} - \mu_1 \frac{\partial^2 \Phi}{\partial \theta^2} \right), \tag{4c}$$

$$(i = 1, 2).$$

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In eq. (2) the superscripts `(s)` are omitted in $\Phi^{(s)}_1$ and $\Psi^{(s)}_1$ for simplicity. This will be done to all other scattered quantities because they occur often. For example, in eq. (9e) below, the symbol $D^{(s)}_1$ will be used instead of $D^{(s)pp}_1$.

In eqs (2) and (3), the four sets of coefficients $A_m, B_m, C_m$ and $D_m$ are to be determined from the boundary conditions at the interface between the cylinder and the matrix. These coefficients have been evaluated (e.g. Pao & Mow 1973).

We further simplified the relevant expressions by eliminating redundant terms such that the final expressions are more concise. This leads to better precision in the numerical computations.

The results are

$$A_m = -i e^{im} \frac{\Delta_{2m}}{\Delta_m},$$

$$B_m = -\frac{2\epsilon_m}{\pi} \frac{\Delta_{2m}}{\Delta_m},$$

$$C_m = -\frac{2\epsilon_m}{\pi} \frac{\Delta_{3m}}{\Delta_m},$$

$$D_m = -\frac{2\epsilon_m}{\pi} \frac{\Delta_{4m}}{\Delta_m},$$

In the figure, $\theta = 0^0$ for $k_c^2a = 0.5, 1.0, 5.0, 10.0, 20.0, 40.0$. The figure shows the directivity patterns of the scattered waves for non-absorbing and absorbing high-velocity solid cylinders excited by an incident plane $P$ wave for six $k_c^2a = va/v_t$ values from 0.5 to 40.0. (a) $P_s(\theta)$ for the scattered compressional wave and (b) $P_p(\theta)$ for the scattered shear wave. Solid and dashed curves refer to non-absorbing and absorbing cylinders, respectively.
where \( \alpha_n, \beta_n, i = 1, 2, 3, 4 \), are listed in Appendix A. When the radius of the cylindrical scatterer is very small in comparison to the wavelength, i.e. when \( k\alpha \ll 1 \), substituting the small-argument approximations of the Bessel and Hankel functions into eqs (5) we obtain the Rayleigh scattering approximation (Bowman et al. 1969).

In Appendix B, we derive the corresponding expansion coefficients in eqs (5) at some special cases: (1) when the cylindrical inclusion is a cavity, (2) when the solid cylinder is rigid, (3) when both the matrix and the cylinder are fluid, (4) when the solid cylinder is still elastic but the matrix is a fluid, and (5) when the cylindrical inclusion is a fluid.

Eqs (1)–(5) hold for plane \( P \)-wave incidence. For comparison, we have derived analogous expressions for the case of cylindrical \( P \)-wave incidence. The analogous equations are very similar and so can be briefly described. Assume the source of the cylindrical \( P \) wave is located on a line parallel to the cylinder axis (Fig. 1, \( r = b \), where \( b \) is the distance of the source from the cylinder axis, \( \theta = \pi \)). The displacement potential of the incident cylindrical wave can then be represented by the series (Faran 1953; Morse & Feshbach 1953) (Bowman et al. 1969).

\[
\Phi_i^{(cyl)}(r) = \frac{H_1^{(1)}(k_1 r)}{|H_1^{(1)}(k_1 b)|} e^{-ik_1 r} \sum_{n=0}^{\infty} \epsilon_n (-1)^n \times H_n^{(1)}(k_1 b) J_m(k_1 r) \cos(m\theta) e^{-im\theta}, \quad r < b. \tag{6}
\]

This has been normalized by \( |H_1^{(1)}(k_1 b)| \). Starting from this expression and following the same steps as those in the above case of plane \( P \)-wave incidence, we find that the potentials of the scattered and internal waves are given by the same
expressions as eqs (2) and (3), but now the corresponding coefficients are

\[ A_m^{2i} = e_m(-1)^{m+i} \frac{H_1^{(1)}(k_{ci} r)}{H_0^{(1)}(k_{ci} b)} \frac{\Delta_{im}}{\Delta_m}, \]  

\[ B_m^{2i} = \frac{2e_m(-1)^{m+i} H_0^{(1)}(k_{ci} r)}{H_0^{(1)}(k_{ci} b)} \frac{\Delta_{2m}}{\Delta_m}, \]  

\[ C_m^{2i} = \frac{2e_m(-1)^{m+i} H_0^{(1)}(k_{ci} r)}{iH_0^{(1)}(k_{ci} b)} \frac{\Delta_{3m}}{\Delta_m}, \]  

\[ D_m^{2i} = \frac{2e_m(-1)^{m+i} H_0^{(1)}(k_{ci} r)}{-iH_0^{(1)}(k_{ci} b)} \frac{\Delta_{4m}}{\Delta_m}, \]

in which \( \Delta_m, \Delta_{im}, i = 1, 2, 3, 4, \) are the same as those in eq. (5) and listed in Appendix A.

In addition to studying the scattering by a perfectly elastic cylinder, we also computed numerically the scattering by a viscoelastic cylinder. For this purpose, it is only necessary to adopt complex values for the wave velocities in the cylinder. Following Müller (1983), we assume that the complex wave velocities are given by

\[ v_{c1}^* = v_{c1} \left[ 1 + \frac{1}{2} \left( \frac{1}{Q_{c}(v_0)} - \frac{1}{Q_{c}(\omega)} \right) \cot \left( \frac{\pi}{2} - \frac{i}{2} \frac{1}{Q_{c}(\omega)} \right) \right], \]  

\[ v_{c2}^* = v_{c2} \left[ 1 + \frac{1}{2} \left( \frac{1}{Q_{c}(v_0)} - \frac{1}{Q_{c}(\omega)} \right) \cot \left( \frac{\pi}{2} - \frac{i}{2} \frac{1}{Q_{c}(\omega)} \right) \right] \]

Figure 3. Same as Fig. 2 except that the cylinders have low velocities.

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Y. Liu, R.-S. Wu and C. F. Ying

\[ Q_c(\omega) = Q_c(\omega_0) \left( \frac{\omega}{\omega_0} \right)^\gamma, \]

\[ Q_s(\omega) = Q_s(\omega_0) \left( \frac{\omega}{\omega_0} \right)^\gamma, \]

in which \( \gamma \) is a constant related to the absorption property of the cylinder. The case \( 0 < \gamma \leq 1 \) is referred to as generalized Maxwell bodies and the case \( -1 \leq \gamma < 0 \) as generalized Kelvin-Voigt bodies; the case \( \gamma = 0 \) represents frequency-independent \( Q \). \( Q_c(\omega_0) \) and \( Q_s(\omega_0) \) are the compression and shear quality factors at a reference frequency \( \omega_0 \). It is evident that the cylinder becomes elastic when \( Q_c(\omega_0) \) and \( Q_s(\omega_0) \) approach \( \infty \). All the expressions from eqs (1)–(5) remain valid for the visco-elastic cylinder if in those equations the \( \nu_2 \)'s and consequently \( Q_s(\omega_0) \) are the \( k_2 \)'s are modiﬁed according to eq. (8).

2.2 Directivity patterns

The directivity patterns of the scattered waves are of interest in practical applications. Substituting the expressions of potential

\[ \theta = 0^0 \]

\[ k_{c2a} = 0.5 \]

\[ k_{c2a} = 1.0 \]

\[ k_{c2a} = 5.0 \]

\[ k_{c2a} = 10.0 \]

\[ k_{c2a} = 20.0 \]

\[ k_{c2a} = 40.0 \]
functions (2a) and (2b) into the expressions of stresses (4c) and (4d) we have

\[
\sigma_{rr} = -\left(\rho_1 c^2 \sum_{m=0}^{\infty} A_m H_m^{(1)}(k_m r)
+ \frac{2\mu_1}{r} \sum_{m=1}^{\infty} \left[m k_{c1} B_m H_m^{(1)}(k_m r) - k_{c1} A_m H_m^{(1)}(k_m r)\right]
+ \frac{2\mu_1}{r^2} \sum_{m=1}^{\infty} (m - m^2)(A_m H_m^{(1)}(k_m r) + B_m H_m^{(1)}(k_m r))\right) \times \cos(m\theta),
\]

(9a)

\[
\sigma_{\theta\theta} = \left(\rho_1 c^2 \sum_{m=1}^{\infty} B_m H_m^{(1)}(k_m r)
+ \frac{2\mu_1}{r} \sum_{m=1}^{\infty} \left[m k_{s1} B_m H_m^{(1)}(k_m r) - k_{s1} A_m H_m^{(1)}(k_m r)\right]
+ \frac{2\mu_1}{r^2} \sum_{m=1}^{\infty} (m - m^2)(A_m H_m^{(1)}(k_m r) + B_m H_m^{(1)}(k_m r))\right) \times \cos(m\theta).
\]

(9b)

Eqs (9a) and (9b) show that radial and tangential stresses include the contributions from both compressional and shear waves.

Figure 4. Directivity patterns for the high-velocity elastic cylinder excited by an incident cylindrical P wave from \( b = 100 \) km. (a) \( D_{cylp1}^p \) for the scattered compressional wave and (b) \( D_{cylp1}^s \) for the scattered shear wave. The results for cylindrical wave incidence are plotted as dashed curves. For comparison, the results for plane P-wave incidence are plotted as solid curves.
stress components. The radiation patterns of the near field with the $1/r^2$ term, the intermediate field with $1/r$ and the far field can be studied using the above equations. The radiation patterns of near and intermediate fields will be discussed in a future paper. In the following we will only discuss the far-field radiation patterns.

We may define the far-field directivity pattern of scattering as the plot exhibiting the angular distribution of the absolute value of the amplitude of the radial or tangential stress at a large distance from the scatterer, i.e. at $kr \gg 1$. The radial and tangential stresses in the far field can be obtained by retaining only the first terms in eqs (9a) and (9b), thus we have

$$\sigma_{r1} = -\rho_1 v^2 \Phi_1, \quad (9c)$$
$$\sigma_{\theta 1} = \rho_1 v^2 \Psi_1. \quad (9d)$$

From the above equations it can be shown that in the present instance the directivity pattern of the scattered compressional wave is (Faran 1951)

$$D_{p1}^r(\theta) = \sum_{m=0}^{\infty} A_m \cos(m\theta), \quad (9e)$$

while that of the scattered shear wave is

$$D_{p1}^s(\theta) = \sum_{m=1}^{\infty} B_m \sin(m\theta) \quad (9f)$$

for the case of $P$-wave incidence. The subscripts $p$ in $D_{p1}^r$ and $D_{p1}^s$ denote the type of incident wave and the superscripts signify the type of the scattered wave.

**Figure 4.** (Continued.)
2.3 Scattering cross-section

The scattering cross-section of a scatterer is usually defined as the ratio of the total energy flow carried outwards by the scattered wave to the energy flow of the incident wave through a normal area that is equal to the cross-sectional area of the scatterer (Ying & Truell 1956; Van de Hulst 1957; Korneev & Johnson 1993). Thus the scattering cross-section of the compressional wave is

\[ Q_{p_1}(\theta) = \frac{1}{k c_1 a} \left| \sum_{m=1}^{\infty} A_m \right|^2, \]

(10a)

that of the shear wave is

\[ Q_{s_1}(\theta) = \frac{1}{k c_1 a} \sum_{m=1}^{\infty} |B_m|^2, \]

(10b)

and that of the total scattered field is

\[ Q_{p_1}(\theta) = Q_{p_1}(\theta) + Q_{s_1}(\theta). \]

(10c)

2.4 Synthetic seismogram

To obtain a seismogram of elastic wave scattering by a solid cylinder, a transient elastic wave source is used rather than the harmonic wave source studied so far. The expression for the wave pulses received at each of the observation stations may be derived from the steady-state solution of the scattered wave given in Section 2.1 in the frequency domain by applying the Fourier transform technique. For example, in our computations it will be assumed that the incident wave is a Ricker wavelet, the time variation of which is

\[ g(t) = e^{-t^2/2f_{\text{max}}^2} \cos[\pi f_{\text{max}}(t - t_0)], \]

(11)

where \( f_{\text{max}} \) is the maximum frequency and \( t_0 \) is the starting time.

The displacement potentials of the scattered \( P \)-wave and \( SV \)-wave pulses can then be expressed, respectively, as

\[ \phi_{p1}(t) = \int_{-\infty}^{\infty} G(\omega) \Phi_{p1}(\omega) e^{-i\omega t} d\omega, \]

(12a)

and

\[ \psi_{p1}(t) = \int_{-\infty}^{\infty} G(\omega) \Psi_{p1}(\omega) e^{-i\omega t} d\omega, \]

(12b)

where \( G(\omega) \) is the frequency spectrum of \( g(t) \) and \( \Phi_{p1}(\omega) \) and \( \Psi_{p1}(\omega) \) are related to the steady-state responses on writing eq. (2) as \( \Phi_1 = \Phi_{p1}(\omega) e^{-i\omega t} \) and \( \Psi_1 = \Psi_{p1}(\omega) e^{-i\omega t} \). According to eqs (4a) and (4b), the radial and azimuthal displacement components of the scattered wave pulses are given by

\[ u_{r1}(t) = -\int_{-\infty}^{\infty} G(\omega) \sum_{m=0}^{\infty} A_m \left[ \frac{m}{r} H_{m}^{(1)}(k c_1 r) - k c_1 H_{m+1}^{(1)}(k c_1 r) \right] \times \cos(m \phi) e^{-i\omega t} d\omega, \]

\[ u_{\phi1}(t) = -\int_{-\infty}^{\infty} G(\omega) \sum_{m=0}^{\infty} \frac{m}{r} A_m H_{m}^{(1)}(k c_1 r) \sin(m \phi) e^{-i\omega t} d\omega, \]

\[ u_{r2}(t) = \int_{-\infty}^{\infty} G(\omega) \sum_{m=1}^{\infty} B_m \left[ \frac{m}{r} H_{m}^{(1)}(k c_1 r) - k c_1 H_{m+1}^{(1)}(k c_1 r) \right] \times \cos(m \phi) e^{-i\omega t} d\omega, \]

(13)

\[ u_{\phi2}(t) = -\int_{-\infty}^{\infty} G(\omega) \sum_{m=1}^{\infty} B_m \left[ \frac{m}{r} H_{m}^{(1)}(k c_1 r) - k c_1 H_{m+1}^{(1)}(k c_1 r) \right] \times \sin(m \phi) e^{-i\omega t} d\omega, \]

(13)

Figure 5. Normalized scattered cross-sections of scattered compressional wave, scattered shear wave and total scattered field for (a) high- and (b) low-velocity cylinders excited by an incident plane \( P \) wave. Solid and dashed lines represent non-absorbing and absorbing cylinders, respectively.
where the \( u_i \)s are functions of \( r, \theta, \) as well as \( t \), and the superscripts \( p \) and \( s \) indicate, respectively, compressional and shear. Since in the following computations the observation stations will be arranged either along the \( x \)-direction or along the \( y \)-direction of the rectangular coordinates, the displacement components observed will be \( u_{y1} \) and \( u_{x1} \). The appropriate displacement components are then

\[
\begin{align*}
u_{y1}^p(t) &= u_{y1}^p \cos \theta - u_{y1}^s \sin \theta, \\
u_{x1}^p(t) &= u_{x1}^p \sin \theta + u_{x1}^s \cos \theta, \\
u_{y1}^s(t) &= u_{y1}^s \cos \theta - u_{y1}^p \sin \theta, \\
u_{x1}^s(t) &= u_{x1}^s \sin \theta + u_{x1}^p \cos \theta.
\end{align*}
\]

It will be noted that the components of the total displacement at any point in medium 1 are

\[
\begin{align*}
u_{y1}(t) &= u_{y1}^p + u_{y1}^s, \\
u_{x1}(t) &= u_{x1}^p + u_{x1}^s.
\end{align*}
\]

It has been assumed that the incident \( P \) wave travels along the +\( y \) direction with a particle displacement \( u_{y1}^0 = u_{x1}^0 \).

3 RESULTS OF THE NUMERICAL COMPUTATION

3.1 Material parameters

In all of the computed results given below, the material parameters of the matrix medium and the inclusion medium...
are chosen according to Table 1. We call the cylinder a ‘high’- (or ‘low’-) velocity cylinder when the velocity in the cylinder is higher (or lower) than that in the matrix. The radius for both ‘high’- and ‘low’-velocity cylinders is $a = 5.0$ km. For viscoelastic cylinders, we take $Q_{p}(o_0) = 20.0$, $Q_{l}(o_0) = 8.9$, $o_0 = 2\pi \text{ Hz}$, $\gamma = 0.2$. The corresponding complex velocities are given by eq. (8).

### Table 1. The material parameters of the matrix and inclusion media.

<table>
<thead>
<tr>
<th>Medium</th>
<th>$v_c$ (km s$^{-1}$)</th>
<th>$v_s$ (km s$^{-1}$)</th>
<th>$\rho$ (kg m$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic matrix</td>
<td>6.0</td>
<td>3.5</td>
<td>2700</td>
</tr>
<tr>
<td>High velocity</td>
<td>7.8</td>
<td>4.5</td>
<td>3200</td>
</tr>
<tr>
<td>Low velocity</td>
<td>4.2</td>
<td>2.5</td>
<td>2400</td>
</tr>
</tbody>
</table>

3.2 Directivity patterns

The program was written in FORTRAN with double precision and run on a SUN workstation. In order to check our Bessel function program we compared our result with that given by Abramowitz & Stegun (1972) for real variables and Briggs & Lowan (1943) for complex variables. The two results are the same to 15 digits.

Fig. 2 shows the directivity patterns for both non-absorbing (solid curves) and absorbing (dashed curves) high-velocity cylinders when the incident wave is a plane $P$ wave. Fig. 2(a) refers to the scattered compressional wave and (b) to the scattered shear wave. Fig. 3 gives the same results for the low-velocity cylinders. In all of these figures, curves are computed for six different values of $k_2a = \omega(k_2/\gamma) = 2\pi/\lambda_2$ (or equivalently of $k_2a = [(c_2/c_1)k_2a]$, where $\lambda_2$ is the wavelength and $\theta$ is computed in steps of $\theta = 0.05^\circ$. The rate of convergence of the series in eqs (9e) and (9f) depends on the values of $k_2a$ and $k_0a$ ($i = 1, 2$) and the change in the refraction indices. The higher the values of $k_2a$ and $k_0a$, the slower the convergence.

Comparing the calculated results for large $k_2a$ (for example, $k_2a = 40.0$) by taking $m = 100$ and $m = 200$, we find that the results are same. In the following numerical computations for radiation patterns, 150 terms are included for large $k_2a$, i.e. $m$ is taken up to 150. For small $k_2a$ we may take smaller $m$ (for example, $m = 30$ for $k_2a = 0.5$) and can obtain satisfactory results because of the fast convergence of series expansions for small $k_2a$. This also avoids the numerical instabilities in the computation of the Hankel functions with small variables and higher orders.

Similar directivity patterns for the same cylinders, but when the incident wave is a cylindrical $P$ wave, have also been computed. The results are presented here only for the non-absorbing high-velocity cylinder for $b = 100$ km; they are given in Fig. 4 as dashed curves, in (a) for the scattered $P$ wave and in (b) for the scattered $SV$ wave. For comparison, analogous results for the plane $P$-wave incidence are simultaneously plotted as solid curves in these figures.

From Figs 2–4, the following may be noted. (1) As $k_2a$ increases, the scattered $P$ wave concentrates in the forward direction. (2) The scattered $SV$ wave is null both in the exact forward direction and in the exact backward direction, as expected. (3) The patterns seem to be slightly more complicated for the scattered $SV$ wave than for the scattered $P$ wave. (4) As $k_2a$ increases, the scattered wave patterns (both $P$ and $SV$) exhibit more complicated structures. (5) Differences exist between the patterns for the incident plane $P$ wave and the incident cylindrical $P$ wave, but they are not very large for large $b/b/a \gg 1$. (6) The effect of viscosity in the cylinder is noticeable for the material parameters chosen. There exist higher values of $D_{pl}(0)$ for the viscoelastic cylinders over those for the elastic cylinders at some instances, especially in the case of Fig. 3(a) when $k_2a = 10$ and 20. This can be explained as follows: the total field is defined as the superposition of the incident and scattered waves, and the resultant field in the shadow of the cylinder has to cancel the incident field. Since the viscoelastic cylinder is more efficient in blocking the incident field, the scattering field has to be larger in this case (Korneev & Johnson 1996).

3.3 Scattering cross-sections

Figs 5(a) and (b) show the scattering cross-sections of compressional wave, shear wave and total scattered field caused
by a plane $P$-wave incidence for a high-velocity inclusion (Fig. 5a) and for a low-velocity inclusion (Fig. 5b). Solid and dashed curves represent non-absorbing and absorbing cylinders, respectively. The curves are calculated in steps of $k_c a = 0.01$ and $m = 150$. The most striking feature of the scattering cross-section curves is the existence of a sequence of maxima and minima. The maxima in the curves are due to a constructive interference of diffracted and transmitted waves, the minima to a destructive interference. The scattering cross-section for a rigid cylinder or cavity does not show such maxima and minima because the transmitted waves will disappear in these two cases. The energy flux is primarily in the scattered compressional wave. The small high-frequency peaks of the scattered cross-sections that appear in a low-velocity non-absorbing cylinder will disappear in a low-velocity absorbing cylinder; this is due to the fact that the peaks correspond to the resonance scattering and are attenuated by the absorbing cylinder (Flax et al. 1981).

3.4 Synthetic seismograms

We consider here the scattering of an elastic wave pulse instead of a continuous harmonic wave. As mentioned earlier, the wave pulse is taken to be a $P$-type Ricker wavelet described by eq. (11) with $f_{\text{max}} = 4$ Hz in our computation, corresponding to a central frequency of $f_0 = 2$ Hz. The geometry relating the incident plane $P$ wave, the scattering cylinder and the array of receivers is shown in Fig. 1, in which the array is assumed to be horizontal, i.e. parallel to the $x$-axis. The array may otherwise be vertical, parallel to the $y$-axis. The $y$- or $x$-displacement components of the scattered $P$-wave or $SV$-wave pulse are computed according to eq. (14), while the components of the total displacement are computed according to eq. (15). In this paper the number of terms retained in eq. (13) is 200 for horizontal ($x$-direction) and vertical ($y$-direction) receiver arrays (in the low-frequency region we may take smaller $m$ because of the fast convergence of series.
Figs 8 and 9 exhibit results analogous to those of Figs 6(c) and 7(c) when the array of receivers is vertical. The array is along the line \( x = 25 \) km and contains six receivers positioned, respectively, at \( y = 10, 20, 30, 40, 50 \) and 60 km. The \( x \)-components of scattering waves in Figs 8 and 9 are multiplied by 3.

From the geometry shown in Fig. 1, it can be expected that a receiver will successively receive wave pulses of different nature that travel along different paths. The wave pulses are composed of the direct-arriving \( P_1 \) signal and scattered signals. These scattered signals consist of two groups, those transmitted through the cylinder and those diffracted on the surface of the cylinder (sometimes known as the 'creeping wave'). The expansion for small \( ka \), this number being more than adequate to render the truncation errors negligible in all cases of interest to us.

In Figs 6 and 7, the array of eight receivers is horizontal, located at an offset distance of \( y = 25 \) km, and the individual receivers are distributed evenly 3 km apart, starting at \( x = 0 \) and extending to \( x = 21 \) km. The material parameters of the cylinders are listed in Section 3.1. In Figs 6 and 7, (a) shows the displacement components \( u_{pl} \) and \( u_{py} \) of the scattered \( P \) wave, (b) shows the displacement components \( u_{2} \) and \( u_{y} \) of the scattered \( SV \) wave multiplied by 5 and (c) shows the displacement components \( u_{1} \) and \( u_{x} \) of the total wave. Solid and dashed lines represent non-absorbing and absorbing cylinders, respectively.

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Figure 7. (Continued.)

Figure 8. The synthetic seismogram of the total field for the high-velocity cylinders excited by an incident plane \( P \) wave with a vertical array of six receivers located at \( x = 25 \) km, \( y = 10–60 \) km at 10 km intervals. Solid and dashed curves represent non-absorbing and absorbing cylinders, respectively. The \( x \)-component of the scattered wave is multiplied by 3.
transmitted signals may pass directly through the cylinder or may pass through it after being multiply reflected within it (Nussenzeig 1992), which can be analysed using the ray theory. For simplicity we shall focus only on the direct transmitted signals in this paper. A ‘diffracted’ signal results when the incoming P-wave pulse tangentially strikes the cylinder; the reflected P wave will then circle the cylinder on the side of the matrix medium and radiate a P1-waves pulse along the tangential direction (θ = π/2) and an S1-waves pulse along the direction of the critical angle [θ = arcsin(v11/v13)]. On generalizing Keller’s geometrical theory of diffraction, Gilbert (1960) analysed the diffraction of impulsive elastic waves on the surface of a cylindrical cavity. His analysis of the various zones of illumination or shadow very probably apply to a circular cylinder of arbitrary modulus.

We try to identify the individual wave pulses exhibited in Figs 6–9 according to the above analysis. Each wave pulse received is designated by the mode (P or S) that propagates along the paths in the matrix and cylinder. For example, the notation P1S1S1 signifies a transmitted wave pulse that is first the incident P wave in medium 1, then the refracted S wave in medium 2, and finally an emergent S wave, again in medium 1. On the other hand, the notation P1P1S1, for example, signifies a ‘diffracted’ wave pulse that is first the incoming y-directed P1 signal, incident tangentially onto the cylinder surface, that will circumvent the obstacle on the matrix side and will then be radiated at a proper interface point at the critical angle θc = arcsin(v11/v13) as an SV signal into the receiver. Here the hat denotes that the wave is an encircling (or creeping) wave.

The compressional (or shear) diffracted waves from the right- and left-hand sides are superposed in the shadow region and can only be separated in the region far from the shadow region. Generally, the diffracted waves arriving at a later time (corresponding to the diffracted wave from the left-hand side in Fig. 1) have smaller amplitude and are difficult to distinguish. We will thus only mark the diffracted waves from the right-hand side in the seismograms.

In Figs 6–9, some received wave pulses are identified and marked in this manner. The transmitted wave P1S1P1 with much smaller amplitude is not marked in the synthetic seismograms. Korneev & Johnson (1993a,b) gave similar observations in their synthetic seismograms for the case of scattering by an elastic sphere, probably through the same procedure of identification. The small-amplitude vibrations that appear in the later parts of the seismograms for the low-velocity cylinder (Figs 7 and 9) correspond to the multiply reflected waves.

From Figs 6–9 it can be seen that in each figure the first-arriving and major compressional wave pulses are the transmitted P1P2P1 pulse and the diffracted P1P1P1 pulse, the former leading the latter in the case of a high-velocity cylinder (Figs 6a and c and 8) and vice versa in the case of a low-velocity cylinder (Figs 7a and c and 9). The two types of pulses generally partly overlap, except at the receivers sufficiently far to the right in the case of the low-velocity cylinder. The two types can be distinguished in the figures roughly by the presence or absence of the superimposed dashed lines, because the influence from the absorbing property of the cylinder is obvious for the transmitted waves. The diffracted waves propagate along the matrix side of the interface and depend mainly on the material properties of the matrix medium. Only a small portion of the energy of the diffracted wave leaks into the viscoelastic cylinder, so that the effect of the absorbing properties of the viscoelastic cylinder on diffracted P1P1P1 and P1P1S1 waves is small.

Similar comments may be made about the scattered shear wave pulses. The first to arrive is the transmitted shear wave P1P2S1 (Fig. 6b) for the high-velocity cylinder and the diffracted shear wave P1P1S1 (Fig. 7b) for the low-velocity cylinder. There exists strong attenuation for the transmitted shear wave P1S1S1 (Fig. 7b) in the case of an absorbing cylinder because the shear quality factor of an absorbing cylinder is low. Otherwise, the multiply reflected waves inside the cylinders are very weak for a high-velocity cylinder but are conspicuous for a low-velocity cylinder.

### 3.5 SV-wave incidence

The results for a SV-wave incidence are similar to those for P-wave incidence; the corresponding analytical expressions

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**Figure 9.** Same as Fig. 8 except that the cylinders have low velocities.
for plane $SV$-wave incidence are given in Appendix C. In the following we only show briefly the numerical results for the low-velocity cylinders.

Fig. 10 gives the directivity patterns of the scattered $SV$ wave for low-velocity cylinders when the incident wave is a plane $SV$ wave. Solid curves refer to the elastic cylinder, while dashed curves refer to the viscoelastic cylinder. The directivity patterns of the scattered $P$ wave for $SV$-wave incidence are the same as those of the scattered $SV$ wave for $P$-wave incidence (Fig. 3b) and are therefore not repeated here. These are in agreement with the results of the reciprocity relation for elastic wave scattering (Varatharajulu 1977), that is, the directivity patterns that undergo a mode convention are invariant. It can be seen that the major features of the directivity patterns for an incident $SV$ wave are also similar to those for an incident $P$ wave. As $ka$ increases, the scattered $SV$ wave concentrates to the forward direction and exhibits more complicated structures. There exist higher values of $D_{SS}(\theta)$ for the viscoelastic cylinders than for the elastic cylinders in some instances.

Fig. 11 shows the scattering cross-sections of the compressional wave, shear wave and total scattered field for low-velocity cylinders when the incident wave is a plane $SV$ wave. Solid and dashed curves represent non-absorbing and absorbing cylinders, respectively. It can be seen that the features of the curves are similar to those of $P$-wave incidence. The interference of diffracted and transmitted waves causes the maxima and minima in the curves of scattering cross-sections, and the small high-frequency peaks that appear in the non-absorbing

![Figure 10](image.png)

**Figure 10.** Directivity patterns of the scattered $SV$ wave for the low-velocity solid cylinders excited by an incident plane $SV$ wave for six $k_c^2 a = v_a/v_c^2$ values from 0.5 to 40.0. Solid and dashed curves refer to non-absorbing and absorbing cylinders, respectively.

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Normalized scattered cross-sections of the scattered compressional wave, scattered shear wave and total scattered field for low-velocity cylinders excited by an incident plane SV wave. Solid and dashed lines represent non-absorbing and absorbing cylinders, respectively.

Fig. 12 shows the synthetic seismograms of the total field for a vertical receiver array from an incident SV-wave pulse (the parameters are the same as in Fig. 9). Solid and dashed curves represent non-absorbing and absorbing cylinders, respectively. The y-component of the scattered wave is multiplied by 3.

4 CONCLUSIONS

The brief analytical solutions have been derived for elastic wave scattering by solid cylinders caused by a normally incident plane P wave or SV wave. Solutions for cylindrical P-wave incidence and for some special cases of the cylinder and matrix medium have also been derived. Based on the analytical solutions, we numerically computed the directivity patterns, the scattering cross-sections and the seismograms for both elastic and viscoelastic cylinders. Results show that the directivity patterns that undergo a mode conversion are invariant, as ka increases the P-P scattering and SV-SV scattering concentrate to the forward direction, and all scattered waves grow into more complicated structures. There exist a
sequence of maxima and minima in the curves of the scattering cross-sections; the maxima are due to a constructive interference of diffracted and transmitted waves and the minima are due to destructive interference. The small high-frequency peaks of the scattered cross-sections that appear in a low-velocity non-absorbing cylinder correspond to resonance scattering. The total scattered field is mainly the superposition of the geometrically transmitted waves that go through the cylinder and the diffracted waves that propagate on the matrix side of the cylinder interface. The arrival times and the waveforms of these different wave pulses depend on the positions of the observation points and the elastic parameters of the media.

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APPENDIX A: COEFFICIENTS A_m, B_m, C_m AND D_m

The coefficients of expansions A_m, B_m, C_m and D_m in eq. (5) are expressed in terms of Δ_m and Δ_m(i = 1, 2, 3, 4). These expressions are

\[ A_m = \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left\{ (m^2 - 1) [mHP1(1 - mJP2) + mHS1(1 - mJS2) + mJP2(1 - mHS1)] \right. \\
+ mJS2(1 - mHP1) - 1 \left. \right\} + \frac{1}{2} m k_2^2 a^2 (HP1 + HS1)(JP2 + JS2) \]

\[ + \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left\{ 2(m - 1) - \frac{1}{2} k_2^2 a^2 \right\} (mHP1 + mHS1 - 1)JP2JS2 - m(m - 1) k_2^2 a^2 (mJP2 + mJS2 - 1)HP1HS1 \]

\[ + \frac{1}{4} m a^2 (k_2^2 - k_1^2)(JP2 + JS2)(HS1 + HP1) + \frac{1}{2} k_2^2 a^2 (HP1 + HS1) - \frac{1}{2} k_2^2 a^2 (JP2 + JS2) \]

\[ + \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left\{ k_2^2 (mHP1 + mJS2 - 1)HS1 - k_2^2 (mHP1 + mHS1 - 1)JS2 \right\} (JP2 + JS2) \]

\[ + \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left\{ k_2^2 (mHP1 + mJS2 - 1)HS1 - k_2^2 (mHP1 + mHS1 - 1)JS2 \right\} (JP2 + JS2) \]

\[ + \frac{1}{4} k_2^2 a^2 (1 - mJS2 - mJP2)(HP1HS1) \right\} \]
For a cylindrical cavity, the internal field will disappear. Then \( \Delta_{3n} = \Delta_{4n} = 0 \) and

\[
\Delta_{3n} = \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left( m^2 - m - \frac{1}{2} k_{11}^2 a^2 \right) \left( JP1 + HS1 \right) + \frac{1}{2} m k_{12}^2 a^2 (JP1 + HS1) \left( JP2 + HS1 \right) + \frac{1}{4} \mu_2 k_{12}^2 a^2 (JP1 + HS1) \left( JP2 + HS1 \right)
\]

\[
\Delta_{4n} = \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left( m^2 - m - \frac{1}{2} k_{11}^2 a^2 \right) \left( JP1 + HS1 \right) + \frac{1}{2} m k_{12}^2 a^2 (JP1 + HS1) \left( JP2 + HS1 \right) + \frac{1}{4} \mu_2 k_{12}^2 a^2 (JP1 + HS1) \left( JP2 + HS1 \right)
\]

\[
\Delta_{3m} = \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left( m^2 - m - \frac{1}{2} k_{11}^2 a^2 \right) \left( JP1 + HS1 \right) + \frac{1}{2} m k_{12}^2 a^2 (JP1 + HS1) \left( JP2 + HS1 \right) + \frac{1}{4} \mu_2 k_{12}^2 a^2 (JP1 + HS1) \left( JP2 + HS1 \right)
\]

\[
\Delta_{4m} = \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left( m^2 - m - \frac{1}{2} k_{11}^2 a^2 \right) \left( JP1 + HS1 \right) + \frac{1}{2} m k_{12}^2 a^2 (JP1 + HS1) \left( JP2 + HS1 \right) + \frac{1}{4} \mu_2 k_{12}^2 a^2 (JP1 + HS1) \left( JP2 + HS1 \right)
\]

with

\[
\begin{align*}
J_P &= \frac{J_m(k_{11}a)}{k_{11}a H_{m+1}^{(1)}(k_{11}a)}, & H_P &= \frac{H_{m+1}^{(1)}(k_{11}a)}{k_{11}a H_{m+1}^{(1)}(k_{11}a)}, & H_{m+1}^{(1)} &= \frac{H_{m+1}^{(1)}(k_{11}a)}{k_{11}a H_{m+1}^{(1)}(k_{11}a)}.
\end{align*}
\]

**APPENDIX B: SPECIAL CASES**

**B1 Cylindrical cavity**

For a cylindrical cavity, the internal field will disappear. Then \( \Delta_{3m} = \Delta_{4m} = 0 \) and

\[
\begin{align*}
\Delta_{3n} &= \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left( m^2 - m - \frac{1}{2} k_{11}^2 a^2 \right) \left( JP1 + HS1 \right) + \frac{1}{2} m k_{12}^2 a^2 (JP1 + HS1) + m^2 - 1, \\
\Delta_{4n} &= \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left( m^2 - m - \frac{1}{2} k_{11}^2 a^2 \right) \left( JP1 + HS1 \right) + \frac{1}{2} m k_{12}^2 a^2 (JP1 + HS1) + m^2 - 1.
\end{align*}
\]

\[
\begin{align*}
\Delta_{3m} &= \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left( m^2 - m - \frac{1}{2} k_{11}^2 a^2 \right) \left( JP1 + HS1 \right) + \frac{1}{2} m k_{12}^2 a^2 (JP1 + HS1) + m^2 - 1, \\
\Delta_{4m} &= \left( \frac{\mu_2}{\mu_1} - 1 \right)^2 \left( m^2 - m - \frac{1}{2} k_{11}^2 a^2 \right) \left( JP1 + HS1 \right) + \frac{1}{2} m k_{12}^2 a^2 (JP1 + HS1) + m^2 - 1.
\end{align*}
\]

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B2 Rigid cylinder

For a rigid cylinder, the internal field will again disappear. Then $\Delta_{3m} = \Delta_{4m} = 0$ and

$$\Delta_m = mHP1 + mHS1 - 1,$$

$$\Delta_{3m} = (mJP1 + mHS1 - 1) \frac{J_{m+1}(k_{c1}a)}{H_{m+1}^{(1)}(k_{c1}a)},$$

$$\Delta_{2m} = \frac{-m}{k_{c1}k_{s1}a^2} \frac{H_{m+1}^{(1)}(k_{c1}a)}{H_{m+1}^{(1)}(k_{c1}a)}.$$ 

(B2)

B3 Fluid cylinder in a fluid

If the shear moduli $\mu_i$ are equal to zero both inside and outside the cylinder, the solution becomes that of acoustic wave scattering by a fluid cylinder. Then $\Delta_{2m} = \Delta_{4m}$ and

$$\Delta_m = \frac{\rho_2}{\rho_1} (1 - mHP1)JP2 + (mJP2 - 1)HP1,$$

$$\Delta_{3m} = \left[ \frac{\rho_2}{\rho_1} (1 - mJP1)JP2 + (mJP2 - 1)JP1 \right] \frac{J_{m+1}(k_{c1}a)}{H_{m+1}^{(1)}(k_{c1}a)},$$

$$\Delta_{3m} = \frac{1}{k_{c1}k_{s1}a^2} \frac{H_{m+1}^{(1)}(k_{c1}a)}{J_{m+1}(k_{c1}a)}.$$ 

(B3)

B4 Elastic cylinder in a fluid

If the shear modulus $\mu_1$ is zero outside the cylinder, the scattered shear wave will disappear and our result reduces to that of the scattering of an acoustic wave by an elastic cylinder (Faran 1951; Flax et al. 1981). In this case, $\Delta_{2m} = 0$ and

$$\Delta_m = 2\frac{\mu_2}{\rho_1} \left[ (m^2 - m - \frac{1}{2} k_{s2}^2 a^2) JS2JP2 - \left( m^2 - m - \frac{1}{2} k_{s2}^2 a^2 \right) (JP2 + JS2 + (m^2 - 1)) \right] (mHP1 - 1)$$

$$+ \omega^2 a^2 \left[ (m^2 - m - \frac{1}{2} k_{s2}^2 a^2) JS2JP2 - \left( m^2 - m - \frac{1}{2} k_{s2}^2 a^2 \right) (JP2 + JS2 + (m^2 - 1)) \right] (mJP1 - 1)$$

$$\Delta_{4m} = \left[ 2\frac{\mu_2}{\rho_1} \left[ \frac{1}{2} k_{s2}^2 a^2 \right] \left[ 2(m^2 - m - \frac{1}{2} k_{s2}^2 a^2) JS2JP2 - \left( m^2 - m - \frac{1}{2} k_{s2}^2 a^2 \right) (JP2 + JS2 + (m^2 - 1)) \right] (mJP1 - 1)$$

$$+ \omega^2 a^2 \left[ \frac{1}{2} k_{s2}^2 a^2 (mJP2 - 1) JS2 - (mJS2 + mJP2 - 1) \right] J_{m+1}(k_{c1}a),$$

$$\Delta_{3m} = \omega^2 a^2 k_{c1}k_{s1}a^2 \frac{H_{m+1}^{(1)}(k_{c1}a)}{J_{m+1}(k_{c1}a)} JS2 - 1$$

$$\Delta_{4m} = \omega^2 a^2 \frac{(m^2 - m)JP2 - m}{k_{c1}k_{s1}a^2} \frac{H_{m+1}^{(1)}(k_{c1}a)}{J_{m+1}(k_{c1}a)}.$$ 

(B4)

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B Fluid cylinder in an elastic medium

If the shear modulus \( \mu_2 \) is zero inside the cylinder, our result reduces to that of the scattering of an elastic \( P \) wave by a fluid cylinder. In this case, \( \Delta_{in} = 0 \) and

\[
\Delta_m = 2 \frac{\mu_2}{\rho_1} \left( \frac{1}{2} k_{1s}^2 a^2 \left[ 2(m^2 - m) - \frac{1}{2} k_{1s}^2 a^2 \right] H_{1s} + (H_{1s} + (m^2 - 1)) \right) (mJP2 - 1)
+ \omega^2 a^2 \left[ 2(m^2 - m) \left( \frac{1}{2} k_{1s}^2 a^2 \right) (H_{1s} + (m^2 - 1)) \right] P_{in},
\]

\[
\Delta_{1m} = \left[ 2 \frac{\mu_1}{\rho_2} \left( \frac{1}{2} k_{1s}^2 a^2 \left[ 2(m^2 - m) - \frac{1}{2} k_{1s}^2 a^2 \right] \right) P_{in} + \omega^2 a^2 (mJP2 - 1) \right] (mJP2 - 1)
+ \omega^2 a^2 \left[ 2(m^2 - m) \left( \frac{1}{2} k_{1s}^2 a^2 \right) (H_{1s} + (m^2 - 1)) \right] J_{2m},
\]

\[
\Delta_{2m} = \frac{m^2 - m - \frac{1}{2} k_{1s}^2 a^2}{k_{1s} k_{2s} a^2 H_{m+1}(k_{1s} a) H_{m+1}(k_{2s} a)} (mJP2 - 1) + \omega^2 a^2 J_{2m} \]

\[
\Delta_{3m} = - \omega^2 a^2 \frac{\rho_2}{\rho_1} \frac{1}{k_{1s} k_{2s} a^2 H_{m+1}(k_{1s} a) J_{m+1}(k_{2s} a)}.
\]

APPENDIX C: SV-WAVE INCIDENCE

C1 Coefficients \( A_m, B_m, C_m \) and \( D_m \)

For an incident plane \( SV \) wave, the displacement potentials of the incident, scattered and internal waves in the frequency domain can be written as follows (Pao & Mow 1973). The incident \( SV \) wave is

\[
\Psi_{1i} = e^{-i(k_{1s}r - \omega t)} = \sum_{m=0}^{\infty} \epsilon_m m J_m(k_{1s} r) \cos(m\theta) e^{-i\omega t},
\]

the scattered \( SV \) wave is

\[
\psi_{1s} = \sum_{m=0}^{\infty} A_m H_{m}^{(1)}(k_{1s} r) \cos(m\theta) e^{-i\omega t},
\]

the scattered \( P \) wave is

\[
\Phi_{1s} = \sum_{m=1}^{\infty} B_m H_{m}^{(1)}(k_{1s} r) \sin(m\theta) e^{-i\omega t},
\]

the internal \( SV \) wave is

\[
\psi_{2i} = \sum_{m=0}^{\infty} C_m J_m(k_{2s} r) \cos(m\theta) e^{-i\omega t}
\]

and the internal \( P \) wave is

\[
\Phi_{2i} = \sum_{m=1}^{\infty} D_m J_m(k_{2s} r) \sin(m\theta) e^{-i\omega t}.
\]

The coefficients of expansions \( A_m, B_m, C_m \) and \( D_m \) can be derived from boundary conditions and have the same expressions as eqs (5) for a plane \( P \)-wave incidence except for \( \Delta_{in} \) and \( \Delta_m \) instead of \( \Delta_{in} \) and \( \Delta_{in} \), respectively. The expressions for \( \Delta_{in} \) and \( \Delta_{in} \) are
similar to eqs (A1)–(A6) for the case of an incident plane $P$ wave and are as follows:

$$\Delta_m = \Delta_n. \quad \text{(C6)}$$

$$\Delta_{m} = \left[ \left( \frac{\mu_2}{\mu_1} - 1 \right) \right] \left[ (m^2 - 1) \left( m HP_1 (1 - m J P_2) + m J S_1 (1 - m J S_2) + m J P_2 (1 - m J S_1) \right) + m J S_2 (1 - m H P_1 - 1) + \frac{1}{2} m k_2^2 a^2 (H P_1 + J S_1 (J P_2 + J S_2)) \right]$$

$$+ \left( \frac{\mu_2}{\mu_1} - 1 \right) \left[ \frac{1}{2} k_2^2 a^2 \frac{\mu_2}{\mu_1} \left[ 2 m (m - 1) - \frac{1}{2} k_2^2 a^2 \right] (m H P_1 + m J S_1 - 1) J P_2 J S_2 - m (m - 1) k_2^2 a^2 (m J P_2 + m J S_2 - 1) H P_1 J S_1 \right.$$  

$$+ \frac{1}{2} m a^2 (k_2^2 - k_1^2) (J P_2 + J S_2) (J S_1 + H P_1) + \frac{1}{2} k_1^2 a^2 (H P_1 + J S_1) - \frac{1}{2} k_2^2 a^2 \frac{\mu_2}{\mu_1} (J P_2 + J S_2) \right]$$

$$+ \frac{1}{4} k_1^2 a^2 \{ (k_1^2 (m H P_1 + m J S_2 - 1) J S_1 - k_2^2 (m H P_1 + m J S_1 - 1) J S_2) J P_2 + k_1^2 (m J S_1 + m J P_2 - 1) H P_1 J S_2 \}$$

$$+ \frac{1}{4} k_1^2 a^2 (1 - m J S_2 - m J P_2) H P_1 J S_1 \right] J_{m+1}(k_{11} a) \left[ m H P_1 J S_1 (k_{11} a) \right]. \quad \text{(C7)}$$

$$\Delta_{2m} = - \Delta_{2n}, \quad \text{(C8)}$$

$$\Delta_{3m} = \left( \frac{\mu_2}{\mu_1} - 1 \right) \frac{1}{2} k_2^2 a^2 \{ (m^2 - m) [(m H P_1 - 1) J P_2 + (m J P_2 - 1) H P_1] - 1 \}$$

$$- \frac{1}{4} k_1^2 a^2 \left[ \frac{\mu_2}{\mu_1} \frac{k_2^2}{k_1^2} \left( m H P_1 - 1 \right) J P_2 - k_2^2 (m J P_2 - 1) H P_1 \right] \right] \frac{1}{k_{11} k_{22} a^2 H_{m+1}^{11}(1(k_{11} a)) J_{m+1}(1(k_{22} a))}, \quad \text{(C9)}$$

$$\Delta_{4m} = - \left( \frac{\mu_2}{\mu_1} - 1 \right) \frac{1}{2} k_2^2 a^2 \{ (m^2 - m) \left[ (1 - 2 m H P_1) J S_2 + H P_1 \right] - m \}$$

$$- \frac{1}{4} m k_2^2 a^2 \left[ \frac{\mu_2}{\mu_1} \frac{k_2^2}{k_1^2} \right] H P_1 J S_2 \right] \frac{1}{k_{11} k_{22} a^2 H_{m+1}^{11}(1(k_{11} a)) J_{m+1}(1(k_{22} a))}. \quad \text{(C10)}$$

**C2** Cylinder cavity

For a cylindrical cavity, the coefficients of expansions $\Delta_m$ and $\Delta_n$ are similar to eq. (B1). Then $\Delta_{m} = \Delta_{m} = 0$ and

$$\Delta_{m} = \Delta_{n}. \quad \text{(C11)}$$

$$\Delta_{3m} = \left\{ k_{11} a^2 \{ (m^2 - m) - \frac{1}{4} k_{11} a^2 \} H P_1 J S_1 + \left( - m^2 + m + \frac{1}{2} k_{11} a^2 \right) (H P_1 + J S_1) + m^2 - 1 \right\} J_{m+1}(k_{11} a) \left[ m H P_1 J S_1 (k_{11} a) \right]. \quad \text{(C11)}$$

$$\Delta_{2m} = - \Delta_{2n}. \quad \text{(C12)}$$

**C3** Rigid cylinder

For a rigid cylinder, the coefficients of expansions $\Delta_m$ and $\Delta_n$ are similar to eq. (B2). Then $\Delta_{m} = \Delta_{m} = 0$ and

$$\Delta_{m} = \Delta_{n}. \quad \text{(C12)}$$

$$\Delta_{3m} = \left( m H P_1 + m J S_1 - 1 \right) J_{m+1}(k_{11} a) \left[ H_{m+1}^{11}(1(k_{11} a)) \right]. \quad \text{(C12)}$$

$$\Delta_{2m} = - \Delta_{2n}. \quad \text{(C12)}$$

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C4 Fluid cylinder

For a fluid cylinder, the coefficients of expansions $\Delta_m$ and $\Delta_m'$ are similar to eq. (B5). Then $\Delta_{3m} = 0$ and

$$\Delta_m = \Delta_m',$$

$$\Delta_{3m} = \left[ 2 \frac{\mu_1}{\rho_1} \left( \frac{1}{2} k_{11}^2 a^2 \right) JP1JS1 - \left( m^3 - m - \frac{1}{2} k_{11}^2 a^2 \right) (JP1 + JS1) + (m^2 - 1) \right] (mJP2 - 1)$$

$$+ \omega^2 \alpha^2 \left[ \frac{1}{2} k_{11}^2 a^2 (mJP1 - 1) JS1 - (mJS1 + mJP1 - 1) \right] \frac{J_{m+1}(k_{11} a)}{H_{m+1}(k_{11} a)},$$

$$\Delta_{2m} = -\Delta_{2m},$$

$$\Delta_{4m} = \frac{\omega^2 \alpha^2 P_1}{\rho_2} k_{11} k_{22} a^2 H_{m+1}^{(1)}(k_{11} a) J_{m+1}(k_{22} a).$$