Infinite boundary element absorbing boundary for wave propagation simulations

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ABSTRACT
In the boundary element (BE) solution of wave propagation, infinite absorbing elements are introduced to minimize diffractions from truncated edges of models. This leads to a significant simplification and reduction of computational effort, especially for 3-D problems. The infinite BE absorbing boundary condition has a general form for both 2-D and 3-D problems and for both acoustic and elastic cases. Its implementation is facilitated by the introduction of the corresponding shape functions. Numerical experiments illustrate a nearly perfect absorption of unwanted diffractions. The approach overcomes some of the difficulties encountered in conventional absorbing techniques and takes less memory space and less computing time.

INTRODUCTION
In this paper, we develop an absorbing boundary element (BE) technique for wave propagation simulation in two and three dimensions using integral equations based on Green’s theorem. The BE method has been applied successfully for numerical integration in the frequency domain (Fu and Mu, 1994; Fu, 1996) as well as for solving the volume integral (VI) equation via a velocity-weighted wavefield, also formulated in the frequency domain (Fu et al., 1997). Here we give a detailed derivation and further improvement of this boundary condition. Particular attention is paid to the implementation of the technique in practical computations.

With the aid of the free-space Green’s function, the partial differential equation for elastic waves can be transformed into a boundary integral (BI) equation for homogeneous media or a volume integral equation for inhomogeneous media. When these integral equations are employed for calculations of wave propagation, artificial reflections arise at the edges of the domain of computation. To avoid these spurious reflections, the most straightforward method is to extend the domain of computation so that reflected signals may not reach the region of interest within the given time period. Obviously, the approach wastes computational resources. One also could use a damping layer of finite thickness to attenuate artificial reflections (Marfurt, 1984; Cerjan et al., 1985; Shin, 1995). Although this “sponge” technique is implemented easily and has been applied to finite-difference, finite-element, and pseudospectral methods for acoustic and elastic wave computations, a relatively thick artificial boundary is needed around the model for the attenuation of the unwanted reflections to an acceptable level. The extra computational effort often results in unreasonably high costs for large scale 2-D and 3-D models.

The most widely used numerical schemes to get rid of artificial reflections from the boundaries of models are based on the paraxial approximation to the wave equation (Lindman, 1975; Clayton and Engquist, 1977; Reynolds, 1978; Engquist and Majda, 1979). Some authors have made substantial improvements to the absorbing boundary conditions for acoustic and elastic finite-difference calculations in the time domain (Mahrer, 1986; Randall, 1988; Renault and Petersen, 1989; Higdon, 1991; Peng and Toksöz, 1995). Among these, only the first-order boundary condition fits into the BI equation form in the acoustic case because it contains only the scalar potential and its first-order derivative with respect to the outward normal of the boundary. The application of the Clayton and Enquist boundary condition in frequency-domain modeling results in an asymmetric, complex impedance matrix (Pratt and Worthington, 1990) which doubles the computing time for factorization of the complex impedance matrix. This first-order Clayton-Engquist boundary condition in the frequency-domain is known also as the Sommerfeld far-field radiation condition in the Green’s integral equation formulation of wave propagation in exterior regions. To suppress unwanted surface integrals of the BI equation on a sphere at infinity requires that this boundary condition be imposed on...
the far-field behavior of displacements and stresses. Numerical experiments using the first-order boundary condition for the frequency-domain BE modeling have shown that only waves of nearly normal incidence on the boundary are suppressed adequately, whereas unacceptably high reflectivities appear for waves traveling obliquely to the artificial boundary. This level of performance is not admissible for practical applications. Higher-order absorbing boundary conditions developed effectively for finite-difference calculations in the time domain can absorb obliquely incident waves to a great extent but are not convenient to be used in solutions of the BI and VI equations in the frequency domain. Furthermore, artificial boundaries must be placed on unbounded sides of the numerical models, resulting in a dramatic increase in computational effort, particularly for 3-D models.

The boundary conditions developed in the present paper are based on a combination of characteristics of the BI or VI equations with an infinite BE method. Integral equations based on the free-space Green’s function enjoy a unique advantage in defining domains extending to infinity. For solutions of the BI equation by the BE method, artificial boundaries around a numerical model will be replaced by several truncation points for a 2-D problem and truncation edges for a 3-D problem. These points are situated at the exterior ends of the interfaces and are made to recede to infinity so that the surface integrals on a sphere at infinity could be assumed to be null valued. Thus, in the simulation of wave propagation, the annoying problem of reflections from artificial boundaries is transformed into the problem of elimination of diffractions from the endpoints. This leads to major simplifications in analyzing and handling artificial waves and saves a large amount of computational resources. In this paper, an efficient absorbing boundary scheme based on the infinite element method is developed. The adoption of the infinite boundary element at the endpoints of interfaces rather than setting vertical artificial computational boundaries along the two sides of a numerical model. Some limitations and assumptions are important factors in the choice of boundary conditions. For some absorbing boundary conditions, elastic waves are processed by the separation of the incident vector displacement fields into compressional and shear components; the elastic medium is assumed to be homogeneous in the region adjacent to the artificial boundary; the performance of the boundary conditions depends on the Poisson’s ratio of the elastic medium. Numerical simulations have shown that such restrictions appear not to be important for the infinite BE absorbing boundary technique.

BOUNDARY TRANSFORM

Before discussing the infinite BE absorbing boundary technique, we first review the boundary transform of unbounded media for solutions of the integral equations. By means of the boundary transform, one finds that artificial boundary conditions defined around a numerical model are equivalent to the infinite BE boundary condition at endpoints of interfaces plus the radiation conditions at infinity. We summarize some properties of this boundary replacement.

For simplicity, the discussion in this section is applied to a 2-D model consisting of two contiguous homogeneous half-spaces \( z > 0 \) and \( z < 0 \). The interface \( \Gamma_{AB} \) separates these two domains (Figure 1a). The same analysis can be applied to a 3-D case. Because the domains \( \Omega_1 \) and \( \Omega_2 \) are unbounded, artificial boundaries \( \Gamma_0 \) and \( \Gamma_\infty \) are placed at the edges of the domains to form closed surfaces \( \Gamma = \Gamma_0 + \Gamma_{AB} \) and \( \Gamma = \Gamma_\infty + \Gamma_{AB} \). Consider the elastic equation for steady-state wave in \( \Omega_1 \):

\[
\mu \nabla^2 u(r) + (\lambda + \mu) \nabla \cdot u(r) + \rho \omega^2 u(r) = -f(r). \tag{1}
\]

Suppose the source distribution consists of a simple point source at \( s \). With the aid of the free-space Green’s function, equation (1) can be transformed into the following boundary integral equation (Pao and Varatharajulu, 1976) for the displacement \( u_i(r) \) on the boundary \( \Gamma \):

\[
C_{ij}(r)u_j(r) + \int_\Gamma T_{ij}(r, r')u_j(r')\,d\Gamma(r')
\]

\[
= \int_\Gamma U_{ij}(r, r')t_j(r')\,d\Gamma(r') + \rho U_{ij}(r, s)f_j(\omega), \quad (i, j = 1, 2, 3) \tag{2}
\]

where the coefficients \( C_{ij}(r) \) generally depend on the local geometry at \( r \), \( t_j(r') \) are the traction components, and \( U_{ij}(r, s) \) and \( T_{ij}(r, r') \) are the fundamental solutions (Green’s tensors) for displacements and tractions, respectively.

For the domain extending to infinity, the boundary integral on \( \Gamma_\infty \) is assumed to vanish, that is,

\[
\lim_{r \to \infty}\int_{\Gamma_\infty} (U_{ij}t_j - T_{ij}u_j)\,d\Gamma = 0. \tag{3}
\]

Using approximate relations in the fundamental solutions as \( r = |r - r'| \) approaches infinity, we have the following scalar

\[
\begin{align*}
\text{FIG. 1.} \quad & \text{Two adjoined half-space models. (a) Model consisting of the interface } \Gamma_{AB} \text{ and two artificial boundaries } \Gamma_0 \text{ and } \Gamma_\infty. \\
& \text{(b) Model after the boundary transform.}
\end{align*}
\]
absorbing boundary condition on \( \Gamma_\infty \):

\[
\frac{\partial u}{\partial n} = (ik \cos \theta) u \tag{4}
\]

where \( \theta \) is the incident angle measured from the normal to the boundary, \( k \) is the scalar wavenumber, and \( \partial/\partial n \) denotes differentiation with respect to the outward normal. In a similar way, the elastic absorbing boundary condition can be derived as

\[
\begin{align*}
\lim_{r \to \infty} \hat{r} (i k \cos \theta \cdot u - t) \cdot \hat{r} &= 0 \\
\lim_{r \to \infty} \hat{r} \cdot (i k \cos \theta \sin \theta \cdot u - t) &= 0
\end{align*}
\tag{5}
\]

where unit vectors \( \hat{r} \) and \( \hat{t} \) are in the normal and tangential directions, respectively. Let \( k_p = i o p V_p \cos \theta, k_S = i o p V_S \sin \theta, \) \( \vec{r} = (m, n), \) \( u = (u_1, u_2) \) (displacement), and \( t = (t_1, t_2) \) (traction) for the 2-D problem. Then equation (5) can be written as

\[
\begin{bmatrix}
 t_1 \\
 t_2
\end{bmatrix} = \frac{1}{m^2 + n^2} \begin{bmatrix}
 k_p m^2 + k_S n^2 \\
 (k_p - k_S) mn
\end{bmatrix} \begin{bmatrix}
 u_1 \\
 u_2
\end{bmatrix}.
\tag{6}
\]

The same formulas can be obtained for the domain \( \Omega_2 \). Equation (4) also can be obtained by applying the Fourier transform in time to the first-order Clayton-Engquist or Sommerfeld boundary condition. From equation (3), the conditions (4) and (6) are required to be imposed on the far-field behavior of the displacement and stress. Therefore, they are satisfied exactly by any plane wave traveling out of the domains \( \Omega_1 \) and \( \Omega_2 \) at any angle, with no reflection returning. Equation (4), although suited for plane waves incident at any angle, requires computing the incident angle \( \theta \) for each position vector \( \vec{r} \) or making a rational approximation to the term \( \cos \theta \) for each individual Fourier component (Randall, 1988) to get perfect absorption to all real angles of incidence. In addition, we must discretize the artificial boundaries \( \Gamma_{AB} \) and \( \Gamma_{\infty} \) by boundary elements, which increases memory space and computing time, especially for 3-D problems.

The boundary condition (4) is known as the radiation condition (as \( \theta = 0 \)), which implies that outward radiation of energy through a sphere at infinity always should be in the outward direction. Therefore, the artificial boundaries \( \Gamma_{\infty} \) and \( \Gamma_{AB} \) vanish if the endpoints \( A \) and \( B \) extend to infinity. This behavior of \( A \) and \( B \) extending to infinity can be approximated by infinite boundary elements. That is to say that the artificial absorbing boundary conditions on \( \Gamma_{AB} \) and \( \Gamma_{\infty} \) are equivalent to the infinite BE absorbing boundary conditions at \( A \) and \( B \) under the control of radiation conditions. The problem of reducing reflections from the artificial boundaries \( \Gamma_{\infty} \) and \( \Gamma_{AB} \) is transferred to the problem of minimizing the diffractions from the artificial endpoints \( A \) and \( B \). The model of Figure 1a is transferred to the model of Figure 1b. Figure 2a shows a two-layer model with its uppermost boundary as the free surface. The dashed lines denote the artificial absorbing boundaries which, according to the boundary transform, can be replaced by four artificial endpoints \( A, B, C, \) and \( D \) in Figure 2b. These points are expected to approach infinity by using infinite boundary elements.

This boundary transform results in significant simplification in dealing with the absorbing boundary problem. The most important advantage is the elimination of the need for artificial absorbing boundaries and replacement of them by several points. As a result, the computational resources are saved so that large 2-D and 3-D models become easy to solve.

### Infinite Boundary Element Absorbing Boundary

In this section, we give formulas for infinite BE absorbing boundary conditions to approximate the behavior of endpoints of a computational interface in two and three dimensions. In the interior, the integral equation can be discretized using boundary elements. The global matrix equation based on the two equations then can be solved to obtain displacements and stresses at all grid points on all interfaces.

First consider the infinite BE approach for 2-D problems solved in the frequency-domain BE representation. In this case, the spatial dimension of the problem is reduced by one. For simplicity, we take the model of Figure 1b as an example. The two homogeneous subdomains, \( \Omega_1 \) and \( \Omega_2 \), have an interface \( \Gamma_{AB} \) defined by \( A < x < B \). The interface \( \Gamma_{AB} \) can be approximated by linear elements shown in Figure 3a, where \( r_1 \) and \( r_2 \) are the node position vectors at the two ends of an element. The local coordinate \( \xi \) varies from \(-1\) to \(1\). The shape functions are defined by

\[
\begin{align*}
N_1(\xi) &= (1 - \xi)/2 \\
N_2(\xi) &= (1 + \xi)/2,
\end{align*}
\tag{7}
\]

which give a linear variation (shown in Figure 3b) from \(-1\) to \(1\) in the element. The coordinates, displacements and stresses at a point in the element can be written respectively as

\[
\begin{align*}
u(\xi) &= N_1(\xi) u_1 + N_2(\xi) u_2 \\
u(\xi) &= N_1(\xi) u_1 + N_2(\xi) u_2
\end{align*}
\tag{8}
\]

![Fig. 2. Two-layer models. (a) Model consisting of the free-surface \( \Gamma_{AB} \), the interface \( \Gamma_{CD} \), and two artificial boundaries denoted by dashed lines. (b) Model after the boundary transform.](image)
The integral equation can be discretized by equation (8) at all nodes on \( \Gamma_{AB} \) for \( \Omega_1 \) and \( \Omega_2 \), respectively, and then all discrete equations are assembled under the continuity of displacement and continuity of stress across interface \( \Gamma_{AB} \) into a global matrix equation. The matrix is solved to obtain displacements and stresses at all nodes on \( \Gamma_{AB} \). The seismic response at any location can be computed through numerical integration on \( \Gamma_{AB} \). However, the endpoints \( A \) and \( B \) will yield strong diffractions in the computed seismic response and will need special treatment. An efficient way is to extend them to infinity by using an infinite boundary element.

First consider the endpoint at \( x = A \). For this case, one can set an infinite boundary element stretching from point \( A \) to infinity, as shown in Figure 4a, where \( r_0 \) is a position vector at \( A \), and \( r_1 \) is a reference position vector. The coordinates at a point in the infinite boundary element can be denoted by

\[
\mathbf{r}(\xi) = N_1^\infty(\xi)\mathbf{r}_1 + N_2^\infty(\xi)\mathbf{r}_0
\]

where the infinite shape functions \( N_1^\infty(\xi) \) and \( N_2^\infty(\xi) \) can be defined as

\[
\begin{align*}
N_1^\infty(\xi) &= (1 - \xi)/(1 + \xi) \\
N_2^\infty(\xi) &= 2\xi/(1 + \xi).
\end{align*}
\]

(9)

Figure 4b illustrates these infinite shape functions with a dramatically nonlinear change from 0.0 to \(-1\). Apparently, from equation (10), one has \( \mathbf{r}(\xi) \rightarrow \mathbf{r}_0 \) at \( \xi = 1 \), \( \mathbf{r}(\xi) \rightarrow \mathbf{r}_1 \) at \( \xi = 0 \), and \( \mathbf{r}(\xi) \rightarrow \infty \) at \( \xi = -1 \). From the preceding discussion, the unknowns (displacement and stress) at infinity are assumed to be null valued. Therefore, they vary in the form of a damping exponential in the infinite element. For the infinite boundary element shown in Figure 4a, the displacement and stress are assumed to vary from node \( r_0 \) as follows:

\[
\begin{align*}
\mathbf{u}(\xi) &= \mathbf{u}(r_0)f_u(\xi, k) \\
\mathbf{t}(\xi) &= \mathbf{t}(r_0)f_t(\xi, k)
\end{align*}
\]

(11)

where \( f \) is a decay function with respect to the local coordinate \( \xi \) and the frequencies. They specify the variation of the unknowns in the infinite direction from their values at node \( r_0 \).

Some empirical methods can be used to choose suitable damping functions for a specific problem. Numerical investigation showed that the optimal damping functions for a 2-D problem can be defined as

\[
\begin{align*}
f_u(\xi, k) &= |H_0^{(1)}(kr(\xi))| \\
f_t(\xi, k) &= |H_1^{(1)}(kr(\xi))|
\end{align*}
\]

(12)

where \( H_0^{(1)}(\cdot) \) and \( H_1^{(1)}(\cdot) \) are Hankel’s functions of the first kind with zeroth order and first order, respectively, and \( r(\xi) = |\mathbf{r}_0 - \mathbf{r}(\xi)| \). Figure 4c shows the decay performance of \( f_u(\xi, k) \) in the infinite boundary element for four frequencies. Different frequencies correspond to different curves. The higher the frequencies, the faster the dampings will be.

Next consider the infinite boundary element for the endpoint at \( x = B \). The infinite boundary element in this case is designed as Figure 5a. The element direction is the positive \( x \)-direction. The reversal of coordinate direction means that the infinite shape functions need to be modified as

\[
\begin{align*}
N_1^\infty(\xi) &= -2\xi/(1 - \xi) \\
N_2^\infty(\xi) &= (1 + \xi)/(1 - \xi).
\end{align*}
\]

(13)

A description of the performance of (13) is given in Figure 5b. The formulas for coordinates, displacements, and stresses at a point in the infinite boundary element have the same form as equations (9) and (11) but with different decay functions in Figure 5c because of the reverse element direction.

The formulas in this section also can be applied to 3-D problems with minor modification. A quadrilateral infinite boundary element for calculations of the 3-D BI equation is shown.

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**Fig. 4.** Infinite boundary element for the artificially ending point \( A \) in Figure 1b. (a) The geometry of the element. (b) The infinite shape functions. (c) The damping functions for the unknowns (displacement and stress) in the infinite boundary element.

**Fig. 5.** Infinite boundary element for the artificially ending point \( B \) in Figure 1b. (a) The geometry of the element. (b) The infinite shape functions. (c) The damping functions for the unknowns (displacement and stress) in the infinite boundary element.
in Figure 6, where A and B are neighboring nodes at the end edge of a curved interface extending to infinity. The position vector at a point in the element can be written as

\[ \mathbf{r}(\xi) = \sum_{i=1}^{4} r_i N_i^\infty(\xi) \]  

where \( N_i^\infty(\xi) \) are 2-D infinite shape functions which can be constructed by

\[
\begin{align*}
N_1^\infty(\xi) &= (1 - \xi_1)/(\xi_1 - 1) \\
N_2^\infty(\xi) &= (1 + \xi_1)(1 - \xi_2)/(2(1 - \xi_1)) \\
N_3^\infty(\xi) &= (1 + \xi_1)(1 + \xi_2)/(2(1 - \xi_1)) \\
N_4^\infty(\xi) &= (1 + \xi_1)(1 + \xi_2)/(\xi_1 - 1).
\end{align*}
\]  

Obviously, the shape functions satisfy \( \sum_{i=1}^{4} N_i^\infty(\xi) = 1 \), and \( \mathbf{r}(\xi) = \infty \) at \( \xi_1 = 1 \). The unknowns in the infinite boundary element vary from the edge AB in the same form as equation (11); the only change needed is a substitution of Hankel’s function (12) by exponential functions related to 3-D wave propagation.

**NUMERICAL RESULTS**

We give several examples to show the effectiveness of the above infinite BE algorithm for the absorbing boundary. In these examples, we simulate acoustic and elastic wave propagation in an unbounded stratified medium consisting of homogeneous layers separated by arbitrarily curved interfaces. The computational model can include a free surface, large velocity contrasts between layers, and large values of \( v_p/v_s \). The numerical implementation for these examples is performed again in the frequency domain (Fu and Mu, 1994; Fu, 1996). Each interface is discretized into a finite number of boundary elements, and infinite boundary elements are set at the endpoints of each interface. The boundary integral equation is approximated for all the boundary elements, including the infinite boundary elements, and then transformed into a simultaneous system of linear equations by use of the conditions of continuity. Once the linear system is solved, the displacements at any observation point are calculated explicitly from the displacements and stresses on the interfaces.

To test the validity of the infinite BE absorbing boundary, the simple two-layer model shown in Figure 1b will be used at first. Figure 7a shows synthetic acoustic seismograms obtained without using any absorbing boundary condition. The artificial diffractions from the endpoints A and B are clearly visible. Figure 7b shows acoustic seismograms using an infinite boundary element at B. (c) with infinite boundary elements at A and B, and (d) by extending the interface \( \Gamma_{AB} \) far enough to avoid spurious waves reaching the seismograms.
In the simulation with the infinite BE absorbing boundary technique (in Figures 9c and d), the efficient suppression of these artificial waves is illustrated clearly.

Finally, the infinite BE method is applied to a 2-D complex salt model shown in Figure 10a. The medium is homogeneous piecewise, with the wave velocities (m/s) indicated in the figure. The dimensions of the model are 4000 m horizontally and 900 m vertically. The source is a minimum-phase wavelet with a central frequency of 20 Hz. The synthetic seismograms for CMP records are displayed in Figure 10b, with receivers 1 to 101 located at 0 to 4000 m along the x-axis. It is calculated in the frequency range 0–40 Hz on a PC computer with Intel 166-MHz Pentium Processor. A variable-element dimension technique in the program implementation is adopted to improve the computation speed. The element dimension for each frequency is computed according to the medium velocity and the frequency, and then the model is discretized automatically. From Figure 10b, we see that the artificial diffractions from the ending points at both sides of the model are very weak relative to the interface reflections, demonstrating the efficiency of the infinite boundary element for implementing the absorbing boundary condition.

**CONCLUSIONS**

The absorbing boundary technique developed here is to design simple infinite boundary elements at the endpoints of artificially truncated interfaces extending to infinity. The infinite BE absorbing boundary condition is devised for solving the Green’s function–based integral equations for wave propagation simulation and has a general form for both 2-D and 3-D problems and for both acoustic and elastic cases. Its implementation is based on the frequency-domain BE method. Numerical experiments have shown that the absorbing boundary yields almost perfect absorption of all unwanted waves. The approach overcomes some of the difficulties of conventional absorbing boundary conditions and takes less memory space and less computing time.

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Fig. 9. Synthetic elastic seismograms for the model in Figure 1b with a vertical force. Upper panel: the seismograms of \( u_x \)-component (a) and \( u_z \)-component (b) calculated without absorbing boundary treatment. Lower panel: the seismograms of \( u_x \)-component (c) and \( u_z \)-component (d) calculated with the infinite BE absorbing boundaries.

Fig. 10. (a) The geometry of a 2-D salt model. The velocity unit is m/s. (b) The synthetic acoustic seismograms calculated by the BE method with infinite BE absorbing treatment.


