MEAN FIELD ATTENUATION AND AMPLITUDE ATTENUATION DUE TO WAVE SCATTERING

Ru-Shan WU*

Department of Earth and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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We point out the inadequacy of two widely used approaches of formulating the amplitude attenuation of seismic waves, the formulation of mean-field attenuation and that of scattering coefficient under the single scattering approximation. Using a one-dimensional layered slab, we show that the attenuation of the mean field is merely a statistical effect caused by phase interference among different realizations of the random wave ensemble, and does not represent the amplitude attenuation. We will call the attenuation coefficient of the mean field as the randomization coefficient in order to distinguish it from the amplitude attenuation coefficient.

We also show that the scattering coefficient method leads to the same result as the first order approximation to the renormalized perturbation series (or the bilocar approximation to Dyson's equation) of the mean field. Therefore these two approaches are equivalent to a certain degree. This is also shown by using the one-dimensional layered slab model as an example.

After pointing out the incorrectness of comparing experiments on amplitude attenuation with the mean field formulation, we suggest and discuss some methods of obtaining the mean field in experiments. For one-dimensional problems, the samples must be taken along the whole propagation path in order to use a spatial average to substitute for ensemble average. For a three-dimensional wave field, measurements over a large seismic array can be used to obtain the mean field. The data from Lasa measured by Aki are used to compare with theory; the agreement between them is good. Finally we compare the mean-field attenuation (randomization) and the amplitude attenuation using the back-halfspace-integration approximation introduced by Wu, and compare them with the measured data by Aki. The comparison shows further the inability of the mean-field formulation in dealing with the problem of amplitude attenuation.

Introduction

There is an increasing interest in measuring and modeling scattering and attenuation of short period seismic waves in inhomogeneous media [1–12]. Two main approaches have been used to treat theoretically the attenuation due to scattering by random inhomogeneities. One is the scattering coefficient method [2, 4, 13], which calculates the fractional energy loss using the single scattering approximation, based on Pekeris [14] and Chernov's [15] work. The other is the mean field approach, in which the statistical ensemble-mean of the wave field is taken, and the attenuation of the mean-field derived from the mean-field equation is treated as the averaged amplitude attenuation of seismic waves [16–20]. This method was borrowed from the similar approach in quantum field theory, which has a well-developed formulation and approximation methods. The formulation has been transferred first to wave propagation in random media [21–31], then to seismic waves. However, many authors failed to recognize the differences between the attenuation of the 'mean field' or the 'coherent field' and the attenuation of the actual measured field. In the practice of seismology, because of the smallness of the size of a detector compared with the wavelength, the actually measured wave amplitude is not the
amplitude of the mean field and there has been little work done on obtaining the ‘mean field’. Therefore, the usual comparison between the measured amplitude attenuation and the calculated mean-field attenuation made by some authors cannot produce meaningful results. In fact, the attenuation of the mean field is a different physical quantity from the amplitude attenuation. It is only a statistical effect that measures the rate of randomization or the rate of losing coherence among members of a wave ensemble when passing through a random medium. Therefore, more correctly, the ‘attenuation coefficient’ of the mean field should be called the ‘randomization coefficient’.

Interestingly, these two approaches lead to a similar result (see equation (27) in [25], comparing with Ch. 3 in [15]), even though the starting point and the condition of applicability of them are quite different. The scattering coefficient method assumes the single-scattering approximation and is thus only valid for a small-volume random medium; while the mean-field attenuation is derived under the assumption of an infinite random medium and has included the multiple-scattering effect.

In this paper we will clarify the concept and meaning of mean-field attenuation and prove the equivalence of results from the mean-field formulation and from the scattering coefficient method under certain conditions. Then we will discuss how to obtain the mean field in experiments and compare the data from a large aperture seismic array to the theoretical prediction. Finally we will compare the mean field attenuation with the recently developed formulation for average amplitude attenuation using the back-halfspace-integration approximation.

1. The randomization coefficient — the attenuation coefficient of the mean field

In order to show the meaning of the attenuation coefficient of the mean field, we shall start with a totally statistical method working on a one dimensional model. Consider a plane scalar wave of frequency $\omega$ incident on a random medium composed of $n$ statistically independent homogeneous layers. Every layer has the same velocity mean $C_0$, the same velocity variance, which is small in order to have the medium weakly inhomogeneous, and the same thickness $a$. Thus the random medium is a family of infinite combinations of these layers.

Suppose for a given layer that the actual velocity is $C$ (homogeneous within the layer). We define $S = C_0/C$ as refractive index or relative slowness, so the wave number $k$ in that layer will be

$$ k = \frac{\omega}{C} = \frac{\omega}{C_0} s $$

where $\delta s$ is the fluctuating part of $s$, and $k_0$ is the average wave number. The phase deviation resulting from the traversal of that random layer is $ka - k_0a = k_0a\delta s$. Therefore, the r.m.s. phase deviation will be

$$ \delta \phi_a = (\langle (k_0a\delta s)^2 \rangle)^{1/2} = k_0a\gamma, $$

where $\gamma = (\delta s^2)^{1/2}$ is the r.m.s. deviation of refractive index, which we will call fluctuation index. The phase variance resulting from the traversal of all the $n$ layers is

$$ \langle \delta \phi^2 \rangle = n(k_0\gamma a)^2. $$

Where $\phi_a = k_0na$ is the phase change of the wave passing through the $n$ layer-medium having a uniform wave velocity $C_0$, and $\delta \phi$ is the phase deviation. Suppose $\delta \phi$ is a normal random variable (here $\delta \phi$ is a sum of a large number of independent identically dis-
tributed random variables), then
\[
\langle \psi \rangle = \psi_0 e^{i\phi_0} e^{-(i\delta \phi^2)/2}
\]

(1.5)

(see, e.g. [36], Ch. 14). Setting the propagating distance \( x = na \), in (1.3) and substituting into (1.5) we get
\[
\langle \psi \rangle = \psi_0 e^{i\phi_0} e^{-\gamma^2 k_0^2 ax/2}
\]

(1.6)

where
\[
\psi = 2x^2
\]

is the randomization coefficient or the attenuation coefficient of the mean field. The result (1.7) for the case of \( ka \gg 1 \) is the same as that obtained by other methods, such as that of Uscinski ([33], formula 4–3) by the parabolic approximation of the mean field equation, Tatarski ([27], 29a) by applying the renormalization procedure to the perturbation solutions, Howe ([29], 6.3) by binary interaction approximations (or the first order smoothing method) to the mean field equation and Keller [23], Karal and Keller [25] \(^1\) by a second-order approximation solution to the stochastic equation. In some of these derivations the complexities of mathematics tended to obscure the physical meaning of the results. From our derivation, which did not go into the detailed physical structure or the pertinent differential equations, we can see that the attenuation of the mean field amplitude as expressed by (1.6) is a totally statistical effect, namely the result of phase interference when taking an ensemble average, since we have neglected the energy attenuation (the effect of amplitude attenuation on the decrease of the mean field is much smaller than that of phase randomizations). The amplitude of the mean field will depend on the degree of randomness of phase \( \phi \), which increases with distance of propagation. Therefore, we can label the ‘attenuation coefficient’ of the mean field as the randomization coefficient.

The randomization coefficient \( \nu \) increases with the scale of inhomogeneities \( a \) (correlation length), which further proves intuitively that \( \nu \) is only a measure of the rate of randomization. This is because when the scale of inhomogeneity increases (i.e. when the number of layers decreases for a given \( x \)), the energy loss due to reflections should decrease, contrary to the mean field attenuation (1.7). In the limiting case, when \( a \) grows to infinity, namely the scale of homogeneities becomes greater than the propagation distance, the medium for our model becomes homogeneous, but is still random. Because there is no reflection during propagation in the medium, there should be no amplitude attenuation; however, for randomization of phase, it reaches a maximum. Setting \( a = x \) in (1.6), we get
\[
\langle \psi \rangle = \psi_0 e^{i\phi_0} e^{-\gamma^2 k_0^2 x^2/2},
\]

(1.8)

the same result as obtained by the random Taylor expansion method (see [28], 3.101).

\[1\] See equation (27) of Karal and Keller [25], which is identical to the result obtained by the energy scattering formulation under the single-scattering approximation (e.g. Cernov [15]). We shall show that the two approaches are identical. Therefore the formula (59) of [15] for \( ka \gg 1 \) represents the same result for that case.

2. The randomization coefficient and the scattering coefficient

The scattering coefficient approach computes the attenuation coefficient as the fractional energy loss from the primary wave by scattering per unit distance of propagation under the single scattering approximation. This approximation implies that all the scattered energy is lost from the primary wave field. Therefore, when distance is small, it leads to the same result as the mean field attenuation approach under the bilocal approximation for the renormalized mean field equation (Dyson’s equation). We will show it briefly as follows.

We adopt the smoothing method in the operator form for the derivation of Dyson’s equation and
its bilocal approximation (see [28] IVC). Suppose \( \psi \) is the random field in question. It can be decomposed into a mean part \( \langle \psi \rangle \) and a fluctuating part \( \delta \psi \),

\[
\psi = \langle \psi \rangle + \delta \psi, \tag{2.1}
\]

where \( \delta \psi \) has the property \( \langle \delta \psi \rangle = 0 \). Suppose \( \psi \) satisfies a random linear wave equation

\[
L \psi = (L_0 + L_1) \psi = F, \tag{2.2}
\]

where \( L \) is a linear operator, \( L_1 \) is its random part, \( L_0 \) is the operator for the corresponding non-random medium, and \( F \) is the source term. Defining \( G_0 \) as the Green function operator of \( L_0 \) (namely the inverse operator of \( L_0 \)), we get from (2.2)

\[
\psi = G_0 F - G_0 L_1 \psi. \tag{2.3}
\]

Now we introduce, for convenience, the averaging operator \( P \) such that

\[
\langle \psi \rangle = P \psi, \quad \delta \psi = (I - P) \psi, \tag{2.4}
\]

where \( I \) is the identity operator. Knowing the rules for \( P \) operation,

\[
PL_1 P = 0 \quad \text{(i.e.} \langle L_1 \langle \psi \rangle \rangle = 0), \tag{2.5}
\]

\[
P G_0 = G_0 P,
\]

\[
PF = F,
\]

we get a relation between the mean field \( \langle \psi \rangle \) and the fluctuating field \( \delta \psi \) by applying the averaging operator to (2.3)

\[
\langle \psi \rangle = G_0 F - G_0 P L_1 \delta \psi. \tag{2.6}
\]

If we call \( G_0 F \) the primary field \( \psi^0 \), (2.6) can be written as

\[
\langle \psi \rangle = \psi^0 - G_0 \langle L_1 \delta \psi \rangle. \tag{2.7}
\]

On the other hand, from (2.4) and (2.3) we derive

\[
\delta \psi = -G_0 (I - P) L_1 \langle \psi \rangle + \delta \psi. \tag{2.8}
\]

By formally iterating the above equation, we get

\[
\delta \psi = \sum_{n=1}^{\infty} [-G_0 (I - P) L_1]^n \langle \psi \rangle. \tag{2.9}
\]

Substituting (2.9) into (2.6) results in the Dyson's equation in operator form,

\[
\langle \psi \rangle = G_0 F + G_0 M \langle \psi \rangle, \tag{2.10}
\]

where \( M \) is the so-called mass operator in quantum field theory.

Notice that the Dyson's equation is an exact equation for the mean field. If we take the first order approximation to the mass operator \( M \) in (2.10), we derive the so-called bilocal approximation of Dyson's equation\(^2\) or the first order smoothing approximation of the mean field,

\[
\langle \psi \rangle = G_0 F + G_0 P L_1 G_0 L_1 \langle \psi \rangle. \tag{2.11}
\]

We notice that the first order approximation of the mass operator in (2.10) is equivalent to the first order approximation to the fluctuating field in (2.9)

\[
\delta \psi = -G_0 L_1 \langle \psi \rangle, \tag{2.12}
\]

which is the modified (or local) Born approximation for the fluctuating field. This can be seen by substituting the above approximation (2.12) into (2.6). Since (2.6) is an exact relation, we conclude that the modified Born approximation for the fluctuating field is equivalent to the bilocal approximations for the mean field.

Now we proceed to prove the equivalence of the scattering coefficient approach to the mean field approach under bilocal approximations. The scattered field \( \psi' \), which has a different physical meaning from the fluctuating field, is defined by

\[
\psi = \psi^0 + \psi'. \tag{2.13}
\]

\(^2\) Originally Dyson's equation was derived by Dyson by using the renormalization procedure to the mean perturbation series, and the bilocal approximation was first introduced in the form of Feynman diagrams by Bourret under the name of 'one-fiction approximation' [20, 21] (also see Frisch [28] IV B). Since then similar approximations have been derived under different names, e.g. bilocal approximation [26, 27], second-order approximation to the stochastic equation [25, 24], binary interaction approximation [29, 30], etc.
where $\psi$ is the actual random field, $\psi^0$ is the primary field, or the field when there are no inhomogeneities. The scattered field $\psi'$ is therefore the field caused by the presence of inhomogeneities.

For a short travel distance, the scattered field $\psi'$ can be calculated by the Born approximation,

$$\psi' = -G_0 L_1 \psi^0. \tag{2.14}$$

Because of the smallness of distance, the mean field $\langle \psi \rangle$ differs only slightly from the primary field $\psi^0$. Therefore, the scattered field $\psi'$ calculated by the Born approximation will be nearly equal to the fluctuating field $\delta \psi$ calculated by the modified Born approximation (2.12). It has been shown that, on average, energy can only flow from the mean field to the random field (in accordance with the second law of thermodynamics, [31]). Therefore, the mean field intensity will decrease continuously due to the energy loss by converting to the random field. If the total energy of the scattered field $\psi'$ is considered to be lost, the field attenuation will have the same rate as the mean field because of the approximate equality between (2.12) and (2.14).

From what has been shown above, we know that the single-scattering approximation and the assumption of total loss of scattered energy lead to the equivalence of the energy scattering coefficient and the mean field attenuation. In fact, unlike the fluctuating field $\delta \psi$, the energy of the scattered field can be transferred back to the total field. The scattered energy is not necessarily lost totally. Suppose the primary wave is travelling along the $x$-direction through an $n$ layered medium. For the second layer there is a loss of energy due to single-scattering in the layer but it also gains some energy from the waves scattered by the first layer. For the third layer, the wave will gain scattered energy from single-scattering by the second layer and from double scattering by the first layer. Generally, the $n$th layer will gain scattered energy from layers 1 through $n-1$. In the case of large scale inhomogeneities ($k a \gg 1$), when forward scattering is dominant, the energy loss by scattering can be quite different from the result obtained by the Born approximation, especially in the forward direction. Even when the criterion for the Born approximation is satisfied, namely that the fractional energy loss calculated by single-scattering formula is much smaller than 1, the frequency dependence of the total field attenuation obtained by considering multiple scattering will be different from the single scattering approximation. This can be seen from the frequency dependence of the directivity pattern of the scattered energy. The directivity of the mean intensity of the scattered field for the correlation functions $e^{-\gamma/a}$ is [15]

$$f(\theta) = \frac{1}{(1 + 4 k_0 a^2 \sin^2 \frac{\theta}{2})^{3/2}}. \tag{2.15}$$

where $\theta$ is the angle between incident wave and scattered wave, $k_0$ is the wave number and $a$ is the correlation length. For greater $k_0$, more scattered energy will be concentrated in the forward direction so that scattered energy loss will deviate more seriously from the calculation of the single-scattering approximation.

To make the physical meaning clearer, we will use again the one-dimensional wave propagation model over the sliced random slab. Suppose that the primary wave is

$$\psi^0 = A e^{ik_0 x}, \tag{2.16}$$

and there is no energy loss during propagation (neglecting reflections). In spite of this, as shown below, we will find that there is a scattering loss equal to the mean field attenuation, if we follow the scattering coefficient approach.

The one-dimensional scalar wave equation is

$$(\nabla^2 + k^2)\psi = 0,$$

$$[\nabla^2 + k_0^2 (1 + \delta s)^2]\psi = [\nabla^2 + k_0^2 (1 + 2\delta s + \delta s^2)] \psi$$

$$= 0, \tag{2.17}$$

where $\delta s$ is the fluctuating part of the refractive index (see (1.1)). Neglecting the $\delta s^2$ term and using
(2.13), we get
\[(\nabla^2 + k_0^2)\psi' = -k_0^2a\delta s\psi.\] (2.18)

Now we solve the equation for the scattering field by the first slice. Since \(a\delta s\) is small, we can use the Born approximation
\[(\nabla^2 + k_0^2)\psi' = -2k_0^2a\delta s\psi.\] (2.19)

Knowing the 1-D wave Green function ([34]),
\[g(x, x') = \frac{-i}{2k_0} e^{ik_0|x-x'|},\] (2.20)
and recognizing that
\[L_1 = 2k_0^2a\delta s,\] (2.21)
we obtain the scattered field under the Born approximation
\[\psi'(x) = -\int g(x, x')L_1\psi^0(x') dx'.\]
\[= \int \frac{i}{2k_0} e^{ik_0|x-x'|}(2k_0^2a\delta s)A e^{ik_0x'} dx'.\]
\[= \int_0^a A \frac{i}{2k_0} (2k_0^2a\delta s) e^{ik_0x} dx\]
\[= A e^{ik_0x}k_0a\delta s.\] (2.22)

In obtaining (2.22) we use the fact that \(x \geq a \geq x'\), where \(a\) is the thickness of the layer, in view of the assumption that all the scattered energy goes to the positive \(x\)-direction.

For our special model we can, in fact, use a simpler derivation for scattered waves (2.22). We know that the phase deviation for the wave after passing through a layer is \(k_0a\delta s\), then the resultant field will be
\[\psi = A e^{ik_0x+ik_0a\delta s}\]
\[= \psi^0 + \psi'.\] (2.23)

Therefore, we have
\[\psi' = \psi - \psi^0\]
\[= \psi^0(e^{ik_0a\delta s} - 1).\] (2.24)

When \(k_0a\delta s\) is small,
\[\psi' = \psi^0ik_0a\delta s.\] (2.25)

The mean intensity of the scattered field is
\[\langle |\psi'|^2 \rangle = \langle \psi'\psi'^* \rangle\]
\[= \langle |\psi^0|^2(k_0^2a^2\delta s^2) \rangle\]
\[= |\psi^0|k_0^2a^2\gamma^2.\] (2.26)

where \(\gamma\) is the r.m.s. deviation of refractive index (see 1.2). If we assume that the total energy scattered by single-scattering is lost, the energy loss by scattering of the whole \(n\)-layer slab will be proportional to \(n\langle |\psi'|^2 \rangle\), because of the independence between different layers. Therefore the scattering coefficient defined as the fractional energy loss per unit distance by passing through the random slab will be
\[\frac{1}{na} n\langle |\psi'|^2 \rangle = \frac{nk_0^2a^2\gamma^2}{na}\]
\[= k_0^2a\gamma^2\]
\[= 2\nu,\] (2.27)
where \(\nu\) is the field randomization coefficient, i.e. the attenuation coefficient of the mean field defined by (1.6) and (1.7). We know that the intensity of the mean field will decrease as
\[\langle |\psi|^2 \rangle = |\psi^0|^2 e^{-2\nu x},\] (2.28)
so we can see how the premise of the total loss of single-scattered energy leads to an identity between the scattering coefficient and the attenuation coefficient of the mean field energy.

3. On acquiring experimental data of the mean field with an example from Lasa

A random medium by mathematical definition is an ensemble or a family of innumerable inhomogeneous components. Each component, with a certain probability of existence, differs from others in its detailed structure, but there are some statistical similarities among all components. When we consider wave propagation in a random
medium, we are dealing therefore with a random wave, namely an infinitely large family of waves, each propagating in an inhomogeneous medium with a certain probability. The mean of a random wave field is the ensemble average over the whole wave family. Therefore, statistically, the average should be taken over a great many experiments for different members of the considered random medium under the same condition. For seismic wave propagation in the earth, the ensemble average may be approximated by a spatial average in certain cases in view of the ergodicity, or as considered within the framework of a 'pseudo-random medium' [35, 28]. However, when taking spatial averages, the non-random phase relation between waves at different points must be taken into account. In the case of $ka \gg 1$ (i.e. wavelength much smaller than the scale of inhomogeneity), the spatial averaging, which of course should cover an area spanning many correlation lengths, will be taken over many wavelengths. The average of the waves will diminish due to non-random term phase interference, if no phase corrections for waves at different points have been implemented. Therefore, as we take the spatial average, a phase correction, which corresponds to the phase change caused by propagation from the reference point to the point considered with the average velocity, must be performed.

First we consider the 1-D problem. A scalar monochromatic wave $A e^{ik_0x}$ is incident on and propagates in a random medium along the $x$-direction. Each member of the random medium family is a weakly inhomogeneous medium with a scale of inhomogeneity $a$ (correlation length). We know that the phase deviation of the random wave after passing through a distance $a$ is $k_0a\delta s$, where $\delta s$ is the fluctuation of the refractive index; then the wave becomes

$$\psi(a) = A(a) e^{ik_0a + k_0a\delta s}.$$ 

When the wave travels within the second layer, since $\delta s$ in this layer is uncorrelated with the first layer and is arbitrary, we can consider the two layer medium as being from another member of the random medium family, and so on. Thus for a wave which has traveled a distance $na$,

$$\psi(na) = A(na) \exp\left(ik_0na + i \sum_{j=1}^{n} k_0a\delta s_j\right).$$ (3.1)

For a sufficiently large $n$, assuming the condition for ergodicity is nearly satisfied, we can take the average of these $n$ waves as the approximation of the ensemble average. However, the phase term $e^{ik_0na}$, which has nothing to do with the randomness of the medium, will interfere with the result. Therefore we must perform a phase correction $e^{-ik_0x}$ to eliminate the phase interference. From the above arguments, we conclude that for one-dimensional problems, the mean field at point $x$ can be approximated by

$$\langle \psi(x) \rangle = \frac{1}{x} \int_{0}^{x} \psi(x') e^{-ik_0x'} dx'.$$ (3.2)

Here $\psi(x') = A(x') e^{i\phi(x')}$ is the measured field at point $x' (\leq x)$. Therefore

$$\psi(x') e^{-ik_0x'} = A(x') e^{i\phi(x')},$$

where $\delta \phi(x')$ is the phase deviation from the field of the homogeneous medium.

If the family of $n$ waves for $x = na$ is taken as a random function or a random process, the $\psi(x)$ is a non-stationary process. The decrease of the mean of the process will approximate the 'attenuation' of the mean field, i.e. the randomization of the field. In the simplified model, in which the phase deviation by passing one layer of thickness $a$ can be with equal probability either $\Delta \phi_a$ or $-\Delta \phi_a$, where $\Delta \phi_a$ is a fixed small quantity, the total phase deviation then is a Wiener-Levy process with zero mean and a variance equal to

$$n(\Delta \phi_a)^2 = (x/a)(\Delta \phi_a)^2$$

([36], Ch. 9). We know from (1.5) that the mean of $\psi(x)$ will decrease exponentially with increasing variance of phase deviation, which explains the decrease of mean field with distance $x$. 


The averaging formula for a stationary process

\[ \langle \psi(x_0) \rangle = \frac{1}{2x} \int_{x_0-x}^{x_0+x} \psi(x') e^{-ik_0x'} \, dx' \]

would not have any use here. If the process is stationary, its mean should remain unchanged, and the decreasing nature of the mean field can not be obtained.

For three dimensional isotropic problems, in the case of \( ka \gg 1 \), the measurements on the transverse section (or ones that can be reduced to the transverse section), such as the measurements in a seismic array over many correlation lengths, can be used for the calculation of the mean field. Different ray paths with separations larger than the correlation length can be thought of as different realizations of the random medium. However, no attempt has been made, to my knowledge, in obtaining the experimental value of the mean field of seismic waves and comparing them with theoretical calculations. Some authors simply compare those measurements on amplitude attenuation with the theoretical results for the mean field, or equivalently, with scattering coefficient calculations under the single-scattering approximation. Nevertheless, the coherency measurements using the beam forming method conducted by Mereu and Ojo [37] are closely related to the mean field measurement, though no comparison was made to the mean field formulation.

The records of Lasa, Norsa or any large aperture seismic array can be used to obtain the mean field. Some authors have used these array data in studying the statistical properties of the under-array media [2, 6, 7]. We will use Aki’s data on phase fluctuation across Lasa to infer the mean field attenuation. From the amplitude fluctuation and phase delay fluctuation measurements over the large seismic array, Lasa, consisting of 21 \( \times \) 25 vertical seismographs and covering an area of diameter 200 km, Aki derived a 10 km correlation length for the inhomogeneities, and the velocity fluctuation index

\[ \gamma = \langle (\delta c/c_0)^2 \rangle^{1/2} = 4\% \]

From these parameters, we can calculate the randomization coefficient \( \nu \). To compare the calculated values of \( \nu \) with the experiments, we use the phase fluctuation data given by Aki ([2], table 1). In the table he listed the rms (root mean square) phase fluctuations over the seismic array measured in the frequency range 0.5 Hz–1.5 Hz for 17 events. As we pointed out, when the area of the seismic array is large enough to cover many correlation lengths, the spatial average over the array can approximate the ensemble average. We use only the data for frequencies from 0.5 Hz to 1 Hz. Above 1 Hz there are few events to be averaged. The frequencies used and the corresponding average rms phase fluctuations (over all events), as well as the numbers of events over which the averages were taken, are listed in Table 1.

For \( ka \gg 1 \), forward scattering is dominant. If we neglect the amplitude attenuation in calculating the decrease of the mean field, the randomization coefficient \( \nu \) has a simple relation with the rms phase fluctuation \( \sigma_\phi \). From (1.5) and (1.6), the amplitude of the mean field \( \langle \psi \rangle \) is obtained as

\[ |\langle \psi \rangle| = |\psi_0| e^{-(\delta \phi^2)/2} = |\psi_0| e^{-\nu x}. \]
Therefore, if the average propagation distance $X_0$ in the random medium is known, we can relate the randomization coefficient with the phase variance $\langle \delta \phi^2 \rangle$ as

$$\nu = \frac{\langle \delta \phi^2 \rangle}{2X_0} = \frac{\sigma \phi^2}{2X_0}. \quad (3.4)$$

On the other hand we can calculate $\nu$ from the medium parameter $\gamma$ and $a$ using

$$\nu = \frac{\gamma^2 k_0^2 a}{2}. \quad (1.7)$$

The $\nu$ value calculated from the measured $\sigma \phi$ (assuming $X_0 = 50$ km) and that from the medium parameters are given in Table 1 and are shown in Fig. 1. The square frequency dependence of $\nu$ agrees fairly well with the measured phase fluctuations, though the data are not sufficient for a decisive test.

![Graph](image)

**Fig. 1.** Comparison of $\nu$ randomization coefficient $\nu_\phi$ (attenuation coefficient of mean field) calculated from measured phase fluctuations by Aki at Lasa, Montana and $\nu$, calculated by the mean-field formulation from the medium parameters.

4. **Amplitude attenuation and mean-field attenuation**

We have shown that the mean-field attenuation is mainly a statistical effect due to phase 'interference' among members of a random wave ensemble when the ensemble mean is taken. It has very little connection with amplitude attenuation. A random wave field is a random complex function of position

$$\psi(r) = A(r) e^{i\phi(r)} = A(r) e^{i(\langle \phi(r) \rangle + \delta \phi(r))}, \quad (4.1)$$

where $\psi(r)$ is the random wave, $A(r)$ is its amplitude, and $\phi(r)$ is its phase. Both amplitude $A$ and phase $\phi$ are random variables. Also, $\langle \phi \rangle$ is the ensemble average of $\phi$, and $\delta \phi$ is the fluctuating part. Since phase $\phi$ usually changes much faster than $A$, we can consider $A$ and $\phi$ as linearly independent random variables. Therefore

$$\langle \psi \rangle = \langle A \rangle e^{i\langle \phi \rangle} = \langle A \rangle e^{i\langle \phi \rangle} e^{-\langle (\delta \phi)^2 \rangle/2}. \quad (4.2)$$

The attenuation of the average amplitude $\langle A \rangle$ is usually much slower than the term $e^{-\langle (\delta \phi)^2 \rangle/2}$, so that the mean-field attenuation is determined mainly by the phase-interfering term $e^{-\langle (\delta \phi)^2 \rangle/2}$. If we neglect the attenuation of the average amplitude $\langle A \rangle$, we get

$$\langle \psi \rangle = A_0 e^{-\langle (\delta \phi)^2 \rangle/2} e^{i\langle \phi \rangle}. \quad (4.3)$$

This is the result we obtained in (1.6). Therefore the mean-field attenuation derived from the parabolic approximation or other approximation methods is merely accounted for by the statistical effect due to phase randomization, and has nothing to do with the average amplitude attenuation.

From the above analysis, we know that the information concerning amplitude attenuation cannot be extracted or recovered from the mean field formulation.

In order to derive the average amplitude attenuation, we need to remove the phase influence before taking the ensemble average. The best way is to multiply its complex conjugate to the wave field, which corresponds to performing a phase
correction for every realization of the ensemble before averaging. Thus we get

$$\langle \psi \psi^* \rangle = \langle A e^{i\delta} A e^{-i\delta} \rangle = \langle A^2 \rangle.$$  \hspace{1cm} (4.4)

We must solve the equation for the second moment $\langle \psi \psi^* \rangle$, not for the first moment $\langle \psi \rangle$, in order to describe the average amplitude attenuation of a random wave field.

Recently, Wu [32] has derived the amplitude attenuation from the energy transfer relations within a random slab. The basic idea is the following. The random slab is sliced into layers of thickness $a$ (correlation length) and the Born approximation is used for each slice to calculate the scattered field. To include the multiple scattering effect, only the energy scattered to the back-half-space is considered lost (for the case that the wavelength is much smaller than the correlation length), and the energy correction is done for each successive slice.

The derived amplitude attenuation has a quite different behavior from the mean-field attenuation in the high frequency range. Contrary to the mean-field attenuation, which has a unique frequency-dependence regardless of the form of the correlation function of the inhomogeneities, the frequency dependences of the amplitude attenuation have different forms for different correlation functions. For exponential correlation function, the equivalent inverse quality factor of scattering attenuation $Q_{s}^{-1}$ equals ((22) in [32])

$$Q_{s}^{-1} = \frac{2b}{k_0} = \frac{4\gamma^2k_0^3a^3}{1 + 6k_0^2a^2 + 8k_0^4a^4},$$  \hspace{1cm} (4.5)

where $b$ is the attenuation coefficient of amplitude due to scattering, $k_0$ is the wavenumber for the unperturbed medium, and $a$ is the correlation length. The $Q_{s}^{-1}$ curve has a peak at $k_0a = 1$. In the high frequency range, when $k_0a \gg 1$,

$$Q_{s}^{-1} = \frac{\gamma^2}{2k_0a}. \hspace{1cm} (4.6)$$

Therefore the apparent amplitude attenuation $b$, in the high frequency range can be approximated by

$$b = \frac{\gamma^2}{4a}. \hspace{1cm} (4.7)$$

In the case of Lasa data, taking $\gamma = 0.04$, $a = 10$ km, $v_p = 6$ km/sec, we have $b \approx 4 \times 10^{-5}$/km for frequencies from 0.5 to 1 Hz. Comparing to Fig. 1, we can see that not only is $b$ much smaller than $\nu$, but the frequency dependence of $b$ is also quite different from that of $\nu$. In addition, from (4.7) the attenuation $b$ is nearly inversely proportional to correlation $a$ when $k_0a$ is large. This property is expected for attenuation. In the case when the scale of inhomogeneities is larger than the wavelength, a decrease in the size of inhomogeneities will increase the number of scatterers and therefore increase the scattering energy loss. This is contrary to the mean field attenuation, which decreases with decreasing correlation length $a$ (cf. (1.7) and the related comments).

Fig. 2 shows the comparison between the attenuation measurements and the theoretical prediction by mean field attenuation and by amplitude attenuation. The measurements were made by Aki in Kanto, Japan, using a single-station S-coda-ratio method [3]. In the calculation we took the intrinsic attenuation as $4 \times 10^{-4}$, based on the observation that most of the $Q^{-1}$ curves converge at high frequencies to a certain value, which is assumed to be the intrinsic $Q^{-1}$ [4]. In general the theoretical curve of amplitude attenuation agrees well with the measurements. The discrepancy between the measured data and the prediction by mean-field attenuation further attest to the inadequacy of the mean field formulation for the problem of amplitude attenuation.

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