Scattering characteristics of elastic waves by an elastic heterogeneity

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ABSTRACT

Elastic wave scattering by a general elastic heterogeneity having slightly different density and elastic constants from the surrounding medium is formulated using the equivalent source method and Born approximation. In the low-frequency range (Rayleigh scattering) the scattered field by an arbitrary heterogeneity having an arbitrary variation of density and elastic constants can be equated to a radiation field from a point source composed of a unidirectional force proportional to the density contrast between the heterogeneity and the medium, and a force moment tensor proportional to the contrasts of elastic constant. It is also shown that the scattered field can be decomposed into an “impedance-type” field, which has a main lobe in the backscattering direction and no scattering in the exact forward direction, and a “velocity type” scattered field, which has a main lobe in the forward scattering direction and no scattering in the exact backward direction. For Mie scattering we show that the scattered far field is a product of two factors: (1) elastic Rayleigh scattering of a unit volume, and (2) a scalar wave scattering factor for the parameter variation function of the heterogeneity which we call “volume factor.” For the latter we derive the analytic expressions for a uniform sphere and for a Gaussian heterogeneity. We show the relations between volume factors and the 3-D Fourier transform (or 1-D Fourier transform in the case of spherical symmetry) of the parameter variations of the heterogeneity. The scattering spatial pattern varies depending upon various combinations of density and elastic-constant perturbations. Some examples of scattering pattern are given to show the general characteristics of the elastic wave scattering.

INTRODUCTION

Elastic wave scattering has become a topic of current interest in both general and exploration geophysics, because of its close relation with various kinds of heterogeneities in elastic media. The Earth has been found to be laterally inhomogeneous in every scale, and elastic wave scattering could be the most effective tool to examine these inhomogeneities. In general geophysics, seismic wave scattering by the heterogeneities near the mantle-core boundary of the Earth was proposed by Haddon and Cleary (1974) to interpret the precursors of PKIKP in seismograms (see Doornbos, 1976). The phase and amplitude fluctuation across a large seismic array (such as LASA or NORSAR) were used to estimate the parameters of velocity inhomogeneities under the arrays based on Chernov's theory of wave scattering in random media (Aki, 1973; Capon, 1974; Bertussens et al., 1975). Seismic coda waves from local earthquakes were attributed to backscattering of seismic waves (Aki, 1969; Aki and Chouet, 1975) and attempts have been made to infer the properties of local small-scale heterogeneities from their studies (Aki, 1982; Sato, 1982b; Wu and Aki, 1984). Apparent attenuation caused by seismic wave scattering and its relation with the intrinsic absorption were also discussed (Aki, 1980a, 1980b, 1982; Sato, 1981, 1982a; Wu, 1980, 1982a, 1982b; Richards and Menke, 1983) and the problem is still open. In many of the problems mentioned above the solution is obtained by a scalar wave scattering theory without guarantee of correctness. We need to develop full elastic wave treatments for these problems. In exploration geophysics, following the introduction of shear wave sources and three-component geophones, the need for applying elastic wave scattering to the complex object and structural exploration has become apparent and pressing. Especially in the case of vertical seismic profiling (VSP), where the source and receiver arrangements are favorable for receiving wide-angle reflected or scattered waves and the targets are often complicated, seismic wave scattering has vast possibilities of application. Some VSP experiments have been done to examine the explosion-formed fracture volume and hydrofractures (Turpening, 1984; Turpening and Blackway, 1984).

Therefore, for both general and exploration geophysics, we...
need to advance the elastic wave scattering theory in order to model and understand better the scattering phenomena and also to develop some effective algorithms. Elastic wave scattering by a single inclusion in a homogeneous elastic medium is the basis of more advanced scattering theory, though the topic itself is a rather old one. Elastic wave scattering by an elastic spherical inclusion has been studied by several authors. Ying and Truell (1956) and Yamakawa (1962) treated scattering for plane P-wave incidence and Einspruch et al. (1960) derived the formulas for plane S-wave incidence. (Gubernatis et al. 1977b corrected some errors in earlier works.) In their method the elastic wave equations (a scalar wave equation and a vector wave equation) were decomposed into three scalar wave equations for scalar potentials, which were solved in spherical coordinates, and the boundary conditions were matched to determine the unknown coefficients. The solutions are infinite series, which converge slowly for large inclusions compared with the wavelength. In addition, for a general elastic sphere, a matrix equation must be solved in order to obtain the expanding coefficients for each term of the series. Therefore, there are no explicit expressions for scattered fields and the general characteristics of the spatial scattering pattern have not been exposed. However, in Yamakawa's paper (1962), some results of Rayleigh scattering (when the radius of the sphere is much smaller than the wavelength) for plane P-wave incidence were shown explicitly. Knopoff (1959a, b) also showed some results for a rigid and unmovable sphere. Because of the infinite rigidity and the infinite density of the sphere in his treatment, the scattering pattern derived there is much simpler and not representative for elastic wave scattering.

Besides the complexity of calculation, the eigenfunction expansion method only works for spherical or circular-cylindrical inclusion (see Morse and Feshbach, 1953; Pao and Mao, 1973). For a general case of arbitrary shape and arbitrary parameter variation of the inclusion, only approximate methods are expected.

Instead of solving the partial differential equations, Miles (1960) formulated the scattering problem into an equivalent integral equation using the elastodynamic Green's function derived by Stokes (1849), see also Love, 1944, section 212 and Rayleigh, 1896, section 378) and obtained explicit expressions in the case of Rayleigh scattering using Born approximation. This approach was also used by Haddon and Cleary (1974) for the P-wave scattering near the mantle-core boundary and by Hudson (1977) for the scattered waves in the cods of P. Gubernatis et al. also formulated the scattering problem of a homogeneous inclusion into an integral equation (1977a) and obtained formulas for the case of Born approximation in the whole frequency range (1977b). In this paper, we follow the approach of integral equation and Born approximation, and we generalize the formulation to an elastic heterogeneity of arbitrary shape and arbitrary variation of parameters. We derive the formulas for Rayleigh scattering in terms of equivalent sources having different force and force moments corresponding to the different parameter perturbations. This representation is probably more familiar to geophysicists. In the Mie scattering range (when the wavelength is comparable to the size of inclusion), the scattering pattern can be decomposed into two factors. One is the Rayleigh scattering, the other is the "volume factor," which is a generalization of the "shape factor" of Gubernatis et al. (1977a, b) for uniform inclusions. To demonstrate the scattering characteristics, we give numerical results and plot scattering patterns for a uniform elastic sphere and for a spherically symmetric heterogeneity having Gaussian parameter variations. We also show that the scattered field can be decomposed into an "impedance-type" and a "velocity-type" scattered field with quite different scattering characteristics.

**RAYLEIGH SCATTERING OF ELASTIC WAVES**

Suppose an arbitrary heterogeneity with parameters

\[ \rho(x) = \rho_0 + \delta\rho(x), \]

\[ \lambda(x) = \lambda_0 + \delta\lambda(x), \]

and

\[ \mu(x) = \mu_0 + \delta\mu(x), \]

is situated in an isotropic, homogeneous medium with parameters \( \rho_0, \lambda_0, \) and \( \mu_0, \) where the following conditions are assumed to be satisfied

\[ \delta\rho \ll \rho_0, \quad \delta\lambda \ll \lambda_0, \quad \delta\mu \ll \mu_0. \]

We write the displacement field \( U \) as the sum of "primary wave" \( U^0 \) and "scattered wave" \( U^1 \):

\[ U = U^0 + U^1. \]

The scattered field can be obtained as an integral representation using the body force equivalent method and the Born approximation (Miles, 1960; Haddon and Cleary, 1974; Gubernatis et al., 1977a, b; Aki and Richards, 1980, chap. 13). However, for Rayleigh scattering the scattered field can be equated to the radiation field of an equivalent point source. Since the size of the inclusion is small compared with the wavelength involved, phase differences between the radiation far fields of equivalent body force from different parts of the inclusion can be neglected. Using integration by parts, we get the total single force

\[ F_i = \int Q_i \, dV \approx \omega \delta \rho V^0 U^0_i(x_0), \]

and the force moment tensor of the equivalent point source

\[ M_{ik} = \int Q_i x_k \, dV(x) \]

\[ = - \delta_{ik} \delta\lambda V \cdot U^0_i(x_0) - \delta\mu V [U^0_k(x_0) + U^0_i(x_0)]. \]

From equation (5) it can be seen that the force moment tensor due to \( \delta\lambda V \) has only diagonal elements and all the elements have the same strength, which corresponds to an explosion-type point source; on the other hand, the moment tensor due to \( \delta\mu V \) can have both diagonal and off-diagonal elements. Because of the symmetry with respect to \( i \) and \( k, M_{ik} = M_{ki}, \) the diagonal elements correspond to on-line force couples, while the off-diagonal-element pairs correspond to

Here subscripts before comma denote Cartesian components and subscripts after comma, differentiation with respect to that coordinate. Repeated subscripts imply summation over those subscripts.
torsion-type double couples (we show this in detail in the following).

**Plane P-wave incidence**

When a plane P-wave
\[
U_0^P = \exp \left[ -i\omega (t - x_1/a_0) \right],
\]
\[
U_2^P = U_3^P = 0,
\]
is incident on the inclusion along the x direction (we take \(x_1 = x, x_2 = y, \) and \(x_3 = z\)) where \(a_0\) is the P-wave velocity in the medium, we can calculate the equivalent point forces from equations (4) and (5) as
\[
F_1 = \omega^2 \delta_0 V \exp \left[ -i\omega t \right],
\]
\[
F_2 = F_3 = 0,
\]
and
\[
M = -i \frac{\omega}{a_0} \begin{bmatrix}
\delta_\lambda + 2 \delta \mu & 0 & 0 \\
0 & \delta_\lambda & 0 \\
0 & 0 & \delta_\lambda
\end{bmatrix} e^{-i\omega t}.
\]

Here we took the center of the inclusion at the origin of the coordinates. \(M\) is the moment tensor of the equivalent point source. Therefore, for P-wave incidence the equivalent forces consist of a single force in the incident direction, contributed from the density contrast, and a moment tensor contributed from the contrast of elastic constants. We can see in this case that \(\delta_\lambda\) contributes an isotropic explosion-type source, while \(\delta \mu\) functions as an on-line force couple (Figure 1b).

Knowing the equivalent point source, we can derive the scattered far field (see Aki and Richards, 1980, chap. 4; or Wu, 1984) as
\[
U_i = F_j * G_{ij} + M_{jk} * G_{ij,k}
\]
\[
= \frac{\omega^2 V}{4\pi a_0^3} \frac{1}{r} e^{-i\omega (t - r/a_0)}
\]
\[
\times \left\{ \frac{\delta \rho}{\rho_0} \begin{bmatrix}
\gamma_1 \gamma_1 - \frac{\delta_\lambda}{\lambda_0 + 2 \mu_0} \\
\gamma_2 \\
\gamma_3
\end{bmatrix} + \frac{\omega^2 V}{4\pi a_0^3} \frac{1}{r} e^{-i\omega (t - r/a_0)}
\right\}
\]
\[
\times \left\{ \frac{\delta \rho}{\rho_0} (\delta_{11} - \gamma_1 \gamma_1) - 2 \begin{bmatrix}
\frac{\delta_\rho}{\rho_0} \\
\frac{\delta \mu}{\mu_0}
\end{bmatrix} (\delta_{11} \gamma_1 - \gamma_1 \gamma_1) \right\}.
\]

If we take spherical coordinates having their polar axis in the incident direction \(x_1\) (i.e., in the direction of particle motion) (Figure 1a), we can write the scattered P-wave \(\rho U_r^P\) and S-wave \(\rho U_{s,\text{mer}}^S\) as
\[
\rho U_r^P = \sum_{i=1}^{3} U_i^P \gamma_i
\]
\[
= \frac{V \omega^2}{4\pi a_0^3} \left\{ \frac{\delta \rho}{\rho_0} \cos \theta - \frac{\delta_\lambda}{\lambda_0 + 2 \mu_0} - \frac{2 \delta \mu}{\lambda_0 + 2 \mu_0} \cos^2 \theta \right\}
\]
\[
\times \frac{1}{r} e^{-i\omega (t - r/a_0)},
\]
and
\[
\rho U_{s,\text{mer}}^S = -U_1^S \sin \theta + U_2^S \cos \theta \cos \phi + U_3^S \cos \theta \sin \phi
\]
where the subscript \(r\) stands for the \(r\)-component, and \(mer\) stands for meridian component. Because of the symmetry of the problem with respect to the polar axis, there is no latitudinal component of the S-wave, i.e., \(\rho U_{s,\text{lat}}^S = 0\). This result was obtained by Gubernatis et al. (1977a) by direct integration of the integral representation of the scattered field.

**FIG. 1.** (a) Spherical coordinate system for P-wave incidence, and (b) the scattering patterns for different equivalent forces.
and (11) the scattering will have only a main lobe in the forward direction for P-waves and no backscattering. We called this "velocity-type" scattering.

In general, if \( \delta \lambda / \lambda_0 = \delta \mu / \mu_0 \) and \( \mu = \lambda_0 = \mu_0 \), we can decompose the scattered field into an impedance-type field and a velocity-type scattered field. Since

\[
\frac{Z_{p}}{Z_{0}} = \frac{Z_{p}}{Z_{0}} - \frac{\delta \alpha}{\delta \alpha}
\]

and

\[
\frac{\delta \lambda}{\lambda_0 + 2 \mu_0} = \frac{\delta \mu}{\lambda_0 + 2 \mu_0} = \frac{1}{3} \left( \frac{\delta Z_{p}}{Z_{p}} + \frac{\delta \alpha}{\alpha_0} \right)
\]

where \( Z_{p} \) is the impedance for P-waves, then

\[
R_{e}^{p} = \frac{V \omega^2 e^{-i(\omega - \omega_0)} \delta \lambda \delta \mu}{4\pi a_{0}^2 r}
\]

\[
\times \left[ \cos \theta + \frac{2}{3} \frac{\delta Z_{p}}{Z_{p}} \right]
\]

In a similar way,

\[
R_{e}^{s} = -\frac{V \omega^2 a_{0}^2 e^{-i(\omega - \omega_0)} \delta \lambda \delta \mu}{4\pi a_{0}^2 \beta_{0}^2}
\]

\[
\times \left[ \sin \theta + \frac{2}{3} \frac{\delta Z_{s}}{Z_{s}} \right]
\]

where \( Z_{s} \) is the S-wave impedance, \( Z_{p} = \rho \beta \)

For acoustic scattering, \( \mu = 0 \), and equation (9) agrees with Rayleigh's result (Rayleigh, 1896, section 296, p. 152). However, the existence of shear rigidity increases greatly the complexity of the scattering pattern. Figure 1 gives the spatial patterns of \( R_{e}^{p} \) and \( R_{e}^{s} \) for \( \delta \rho V, \delta \lambda V, \) and \( \delta \mu V \), respectively. Since the decomposition of the spatial pattern of \( R_{e}^{p} \) is not unique, this result is equivalent to that of Miles (1960), but different in representation. In his case, the scattering pattern by \( \delta \mu V \) was expressed as a quadrupole plus an isotropic part, while in our case it is expressed as a dipole. We believe our representation is more natural.

The resultant scattering pattern of an inclusion will exhibit quite different appearances depending upon various combinations of \( \delta \rho, \delta \lambda, \) and \( \delta \mu \). Figures 2 and 3 are examples. Notice that when \( \delta \lambda, \delta \mu, \) and \( \delta \rho \) have the same sign, indicating the inclusion is harder and heavier (or softer and lighter), the p-p scattering always has its maximum in the backscattering direction (Figure 2), because all the scattered field due to \( \delta \lambda, \delta \mu, \) and \( \delta \rho \) are in-phase in backward direction, as seen from equation (9). Also note that, when \( \delta \rho / \rho_0 = \delta \lambda / \lambda_0 = \delta \mu / \mu_0 \), indicating there is no velocity contrast between the inclusion and the medium, i.e. \( \delta \alpha / \alpha_0 = (\delta \rho / \rho_0) + (\delta \lambda / \lambda_0 + 2 \mu_0) \), there will be only one main lobe in the backward scattering direction and no forward scattering (Figure 3). We call this "impedance-type" scattering. When the density perturbation \( \delta \rho \) has the opposite sign to that of the elastic constants \( \delta \lambda, \) and \( \delta \mu, \) i.e., when the inclusion is lighter and harder than the medium (or heavier and softer), the whole scattering pattern will be turned over in such a way as to swap the forward and backward directions. Figure 3 also shows the case of \( \delta \lambda / \lambda_0 = \delta \mu / \mu_0 = -\delta \rho / \rho_0 \). Note that, when there is no impedance contrast between the inclusion and the medium, i.e., when the following quantity vanishes,

\[
\frac{\delta (\rho \mu)}{\rho_0 \mu_0} = \frac{\delta \rho}{\rho_0} + \frac{\delta \lambda}{\lambda_0} + \frac{\delta \mu}{\mu_0}
\]

\[
= \frac{\delta \rho}{\rho_0} + \frac{1}{2} \left( \frac{\delta \rho}{\rho_0} + \frac{\delta \lambda}{\lambda_0} + \frac{\delta \mu}{\mu_0} \right)
\]

\[
= \frac{1}{2} \left( \frac{\delta \rho}{\rho_0} + \frac{\delta \lambda}{\lambda_0} + \frac{\delta \mu}{\mu_0} \right)
\]

the scattering will have only a main lobe in the forward direction for P-waves and no backscattering. We called this "velocity-type" scattering.

In general, if \( \delta \lambda / \lambda_0 = \delta \mu / \mu_0 \) and \( \lambda_0 = \mu_0 \), we can decompose the scattered field into an impedance-type field and a velocity-type scattered field. Since

\[
\frac{\delta \rho}{\rho_0} \frac{\delta \lambda}{\lambda_0} + \frac{\delta \mu}{\mu_0} = \frac{1}{2} \left( \frac{\delta \rho}{\rho_0} + \frac{\delta \lambda}{\lambda_0} + \frac{\delta \mu}{\mu_0} \right)
\]
Plane S-wave incidence

When a plane S-wave

\[ U_i^0 = \exp \left[ -i \omega (t - x_i / \beta_0) \right] \]

and

\[ U_i^0 = U_i^0 = 0 \]

is incident upon the inclusion along the x-direction and having its particle motion in the y-direction (Figure 4a), the equivalent forces can be derived from equations (4) and (5) as

\[ F_1 = F_3 = 0, \]
\[ F_2 = \omega^2 \delta \mu V \exp (-i \omega t), \]

and

\[ M = -i \frac{\omega}{\beta_0} V \begin{bmatrix} 0 & -\delta \mu & 0 \\ \delta \mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{-i \omega t}. \] (15)

namely a single force \( F_2 \) in the direction of particle motion of the incident wave due to density contrast and a double-couple force in the polarization plane due to the contrast of shear rigidity.

The scattered far field can be obtained as

\[ U_i = F_j * G_{ij} + M_{ij} * G_{ij,k} \]

\[ = \frac{\omega^2 \delta \mu V}{4 \pi \rho_0} \left\{ \frac{(\gamma_1 \gamma_2 - \delta_{12})}{\beta_0^2} \frac{1}{r} e^{-i \omega t - r/\rho_0} \right\} \]
\[ + \frac{\omega^2 \delta \mu V}{4 \pi \rho_0} \left\{ \frac{-2 \gamma_1 \gamma_2 + 1}{\alpha_0^2} \frac{1}{r} e^{-i \omega t - r/\rho_0} \right\} \]
\[ + \frac{\omega^2 \delta \mu V}{4 \pi \rho_0} \left\{ \frac{(\gamma_1 \gamma_2 - \delta_{12} \gamma_2 - \delta_{22} \gamma_2)}{\rho_0^2} \frac{1}{r} e^{-i \omega t - r/\rho_0} \right\}. \] (16)

We take the direction of particle motion of the incident field (y-axis) as the polar axis of the spherical coordinates (Figure 4). The scattered P-wave \( U_i^p \) and the scattered S-wave \( U_i^s \) can be written as

\[ s U_i^p = \sum_{i=1}^{3} U_i^p \gamma_i = \frac{V}{4 \pi} \frac{\omega^2}{\alpha_0^2} \left( \frac{\delta \rho}{\rho_0} \gamma_2 - 2 \left( \frac{\beta_0}{\alpha_0} \frac{\delta \mu}{\mu_0} \gamma_1 \gamma_2 \right) \frac{1}{r} e^{-i \omega t - r/\rho_0} \right) \]
\[ = \frac{V}{4 \pi} \frac{\omega^2}{\alpha_0^2} \left( \frac{\delta \rho}{\rho_0} \cos \theta - \left( \frac{\beta_0}{\alpha_0} \frac{\delta \mu}{\mu_0} \sin 2 \theta \sin \phi \right) \frac{1}{r} e^{-i \omega t - r/\rho_0} \right), \] (17)

\[ s U_i^s = -U_i^0 \sin \theta + U_3 \cos \phi \cos \theta + U_1 \sin \phi \cos \theta \]
\[ = \frac{V}{4 \pi} \frac{\omega^2}{\alpha_0^2} \left( \frac{\delta \rho}{\rho_0} \sin \theta + \left( \frac{\beta_0}{\alpha_0} \frac{\delta \mu}{\mu_0} \cos 2 \theta \cos \phi \right) \frac{1}{r} e^{-i \omega t - r/\rho_0} \right), \]
\[ = -\frac{V}{4 \pi} \frac{\omega^2}{\alpha_0^2} \left( \frac{\delta \rho}{\rho_0} \sin \theta + \left( \frac{\beta_0}{\alpha_0} \frac{\delta \mu}{\mu_0} \cos 2 \theta \cos \phi \right) \frac{1}{r} e^{-i \omega t - r/\rho_0} \right), \] (18)

and

\[ s U_i^s = -U_3 \sin \phi + U_1 \cos \phi = \frac{V}{4 \pi} \frac{\omega^2}{\alpha_0^2} \left( \frac{\alpha_0}{\beta_0} \right)^2 \frac{\delta \mu}{\mu_0} \cos \theta \cos \phi \frac{1}{r} e^{-i \omega t - r/\rho_0}. \] (19)
Note that the scattered S-wave due to $\delta \mu V$ is attributed to an equivalent double force couple. For each single couple, if we take the polar axis of the spherical coordinates parallel to the force direction, there will be only meridian components of S-waves. Therefore we can plot the scattering patterns contributed from different equivalent forces as in Figure 4, where we decomposed the pattern due to $\delta \mu V$ into two parts, each of which corresponds to a pattern due to a single couple. The resultant scattering pattern will be the vector sum of those three patterns. Figures 5 to 8 present examples. The S-wave scattering depends only on the contrasts of densities and shear modulus that are physically expected. As in the case of P-wave incidence, the S-S scattering patterns have main lobes in the backscattering direction, if both the density and shear modulus have the same sign. Otherwise the main lobes will turn over to the forward direction (as in Figure 8). In the same manner as for P-wave incidence, there will be no scattering in the forward direction when $\delta \rho/\rho_0 = \delta \mu/\mu_0$, i.e., when the S-wave velocity contrast between the inclusion and the medium vanishes. On the other hand, when $\delta \rho/\rho_0 = -\delta \mu/\mu_0$, or in other words when the shear wave impedance of the inclusion matches that of the medium, there will be no scattering in the backward direction. In x-z plane ($y = 0$) (Figure 6) the meridian component $U_{m}^S$, is the only nonzero component of the scattered waves. The longitudinal component $U_{l}^S$ comes solely from the transverse (with respect to the particle motion of the incident wave) equivalent force couple $M_{12}$. It has two lobes having maxima along the y-axis (Figure 7).

Using equations (12) and (13), the scattered field can be decomposed into an impedance-type field and a velocity-type field as follows,

$$
SU_{m}^P = \frac{V}{4\pi} \frac{\omega^2}{\alpha_0^2} \frac{1}{r} e^{-i\omega t - \frac{r}{\alpha_0}} 
\times \left\{ \frac{\delta Z}{Z_{\alpha_0}} \cos \theta \left[ \frac{\beta_0}{\alpha_0} \sin 20 \sin \phi \right] - \frac{\delta \beta}{\beta_0} \left[ \cos \theta + \left( \frac{\beta_0}{\alpha_0} \sin 20 \sin \phi \right) \right] \right\},
$$

(20)

and

$$
SU_{m}^S = \frac{V}{4\pi} \frac{\omega^2}{\alpha_0^2} \frac{1}{r} e^{-i\omega t - \frac{r}{\alpha_0}} 
\times \left\{ \frac{\delta Z}{Z_{\alpha_0}} \sin \theta + \cos 20 \sin \phi \right\},
$$

(21)

For both P-wave and S-wave incidences, the converted waves (P-S or S-P) have only sidelobes (with respect to the incident direction), while the common-mode scattered waves always have main lobes along the incident direction (either in the forward or backward direction).

Regarding the scattering strength, the scattered S-waves are always stronger than the scattered P-waves (suppose $\delta \lambda \approx \delta \mu$).

FIG. 5. Scattering pattern of Rayleigh scattering in x-y plane for plane S-wave incidence. The upper half is of S-S scattering, the lower half is of S-P scattering. Note that in x-y plane there are only meridian components of scattered S-waves.

FIG. 6. Same as Figure 5, but in x-z plane.

FIG. 7. Same as Figure 5, but in y-z plane.
ELASTIC WAVE SCATTERING OF AN ARBITRARY ELASTIC HETEROGENEITY

In Mie scattering (when the wavelength is comparable to the size of inclusion) the equivalent source of scattering by an inclusion can no longer be regarded as a point source. The phase differences of the incident field at different parts of the inclusion and of the scattered field from different parts of the inclusion can no longer be ignored. Nevertheless, if the total scattered field is still much weaker than the incident field, the Born approximation can still be a useful tool for calculating the scattered field and deriving the scattering characteristics.

Here we formulate the problem for a general arbitrary elastic heterogeneity. The scattered waves can be written using the Born approximation as

$$U_i(x) = \int Q_j(\xi) \cdot G_{ij}(x, \xi) \, dV(\xi), \quad (23)$$

where \(U_i(x)\) is the scattered field at point \(x\), \(G_{ij}\) is the elasticodynamic Green's function, and \(Q_j\) is the equivalent body force (see Aki and Richards, 1980, chap. 4); the integration is over the whole volume. Substituting the expression of \(Q_j\) into equation (23) and integrating out the terms with the gradients of elastic constant using integration by parts, we have

$$\begin{align*}
U_i(x) &= -\int \delta \rho \frac{\partial^2 U_i^0}{\partial t^2} \cdot G_{ij} \, dV \\
&\quad - \int [\delta \lambda (V \cdot U^0) + \delta \mu (U^0_{ij,k} + U^0_{ij,j})] \cdot G_{ij,k} \, dV. \quad (24)
\end{align*}$$

Compared with equations (4) and (5), we recognize that the term inside the square brackets of equation (24) with the factor \(dV\) is the equivalent force moment tensor for the scattering of the elementary volume \(dV(\xi)\) with elastic constant perturbations \(\delta \rho(\xi)\) and \(\delta \lambda(\xi)\) and \(\delta \mu(\xi)\) in the heterogeneity. Therefore equation (24) implies that the scattered field is a superposition of the scattered fields by all the volume elements of the heterogeneity, each of which is of Rayleigh scattering type. Taking the Fraunhofer approximation to \(G_{ij}(\xi)\) (see Aki and Richards, 1980), equation (24) can be further simplified. For plane \(P\)-wave incidence, we have

$$\begin{align*}
S U^P_i(x) &= \frac{\omega^2}{4\pi \sigma_0^2} \frac{1}{r} e^{-i\omega t - \lambda \gamma_1} \\
&\quad \times \left[ \frac{\delta \rho(\xi)}{\rho_0} \gamma_1 + \frac{\delta \lambda(\xi)}{\lambda_0 + 2\mu_0} \gamma_1 \right] \\
&\quad - \frac{2\delta \mu(\xi)}{\lambda_0 + 2\mu_0} \gamma_1 \gamma_1, \quad (25)
\end{align*}$$

In the case of plane \(S\)-wave incidence,

$$\begin{align*}
S U^S_i(x) &= \frac{\omega^2}{4\pi \sigma_0^2} \frac{1}{r} e^{-i\omega t - \lambda \gamma_1} \\
&\quad \times \left[ \frac{\delta \rho(\xi)}{\rho_0} \gamma_1 + \frac{\delta \lambda(\xi)}{\lambda_0 + 2\mu_0} \gamma_1 \gamma_1 \right] \\
&\quad \times \exp \left[ \frac{\omega}{\beta_0} \gamma_1 \frac{\xi_1}{\xi} - i \frac{\omega}{\beta_0} (\hat{s} \cdot \xi) \right] \, dV(\xi), \quad (27)
\end{align*}$$

and

$$\begin{align*}
S U^S_i(x) &= \frac{\omega^2}{4\pi \sigma_0^2} \frac{1}{r} e^{-i\omega t - \lambda \gamma_1} \\
&\quad \times \left[ \frac{\delta \rho(\xi)}{\rho_0} \gamma_1 + \frac{\delta \lambda(\xi)}{\lambda_0 + 2\mu_0} \gamma_1 \gamma_1 \right] \\
&\quad \times \exp \left[ \frac{\omega}{\beta_0} \gamma_1 \frac{\xi_1}{\xi} - i \frac{\omega}{\beta_0} (\hat{s} \cdot \xi) \right] \, dV(\xi), \quad (28)
\end{align*}$$

Suppose \(\delta \rho, \delta \lambda,\) and \(\delta \mu\) have the same form of spatial variation. We introduce the parameter variation function \(P(\xi)\) such that

$$\begin{align*}
\delta \rho(\xi) &= \delta \rho_0 \, P(\xi), \\
\delta \lambda(\xi) &= \delta \lambda_0 \, P(\xi), \quad (29)
\end{align*}$$

and

$$\delta \mu(\xi) = \delta \mu_0 \, P(\xi).$$
where \( \delta \rho_0, \delta \lambda_0, \) and \( \delta \mu_0 \) are the parameter perturbations at the center of the heterogeneity and satisfy

\[
\delta \rho_0 \int_V P(\xi) \, dV(\xi) = \delta \rho V, \quad \text{etc.}
\]

Putting equation (29) into equation (25) yields

\[
\rho U^P(x) = \frac{\omega^2}{4\pi \rho_0} \frac{1}{r} e^{-i\omega t - r/\rho_0} \times \left[ \delta \rho_0 \frac{\gamma_1 \gamma_2}{\lambda_0 + 2\mu_0} + \frac{\delta \lambda_0 \gamma_2}{\lambda_0 + 2\mu_0} - \frac{\delta \mu_0}{\lambda_0 + 2\mu_0} \gamma_1 \gamma_2 \right] \times \int_V P(\xi) \exp \left[ i \frac{\omega}{a_0} \zeta_1 - i \frac{\omega}{a_0} (\hat{x} \cdot \xi) \right] dV(\xi).
\]

We recognize that the formula is the same as the Rayleigh scattering with the volume \( V \) replaced by a factor

\[
\Theta_1(\hat{x}) = \int_V P(\xi) \exp \left[ i \frac{\omega}{a_0} \zeta_1 - i \frac{\omega}{a_0} (\hat{x} \cdot \xi) \right] dV(\xi).
\]

We call \( \Theta_1(\hat{x}) \) the "volume factor." In the case of a uniform inclusion \( P(\xi) = 1 \), it becomes the "shape factor" of Gubernatis et al. (1977b). In this way we can rewrite equations (25) through (28) as

\[
\rho U^P(x) = \frac{1}{r} e^{-i\omega t - r/\rho_0} E_1(\hat{x}) \Theta_1(\hat{x}),
\]

\[
\rho U^S(x) = \frac{1}{r} e^{-i\omega t - r/\rho_0} E_2(\hat{x}) \Theta_2(\hat{x}),
\]

\[
\rho U^T(x) = \frac{1}{r} e^{-i\omega t - r/\rho_0} E_3(\hat{x}) \Theta_3(\hat{x}),
\]

and

\[
\rho U^S(x) = \frac{1}{r} e^{-i\omega t - r/\rho_0} E_4(\hat{x}) \Theta_4(\hat{x}).
\]

where \( E_1-E_4 \) are the elastic wave Rayleigh scattering factors for a unit volume, and \( \Theta_1-\Theta_4 \) are the corresponding volume factors which are the scalar wave scattering patterns

\[
\Theta_4(\hat{x}) = \int_V P(\xi) \exp \left[ i \omega (\hat{x} \cdot \xi) \right] dV(\xi).
\]

In equation 34 we have

\[
\begin{align*}
S_1 &= \frac{1}{a_0} \hat{s}_1 - \frac{1}{\beta_0} \hat{s}, \\
S_2 &= |S_2| = \frac{1}{a_0} \sqrt{\left(1 \frac{1}{a_0}\right)^2 + \left(1 \frac{1}{\beta_0}\right)^2 - \frac{2}{\mu_0} \cos \theta}, \\
S_3 &= \frac{1}{\beta_0} \hat{s}_3 - \frac{1}{a_0} \hat{s}, \\
S_4 &= S_2, \\
S_5 &= \frac{1}{\beta_0} \hat{s}_5 - \frac{1}{a_0} \hat{s},
\end{align*}
\]

and

\[
S_4 = \frac{1}{\beta_0} 2 \sin \frac{\theta}{2}.
\]

where \( \hat{x}_i \) is the unit vector in \( x_i \)-direction, i.e., the incident direction. \( S_1, S_4 \) are the exchange slowness vectors (Figure 9), and \( \theta \) is the scattering angle. Note that equation (34) is in the form of a 3-D spatial Fourier transform. Putting \( K_n = \omega S_n \), we have

\[
\Theta_n(\hat{x}) = \int_V P(\xi) \exp \left[ i (K_n \cdot \xi) \right] dV(\xi) = \hat{P}(K_n).
\]

where \( \hat{P}(K_n) \) is the 3-D Fourier transform of \( P(\xi) \). If the parameter's spatial variation is spherically symmetrical, i.e., \( P(\xi) \) is only the function of \( |\xi| = r \), we introduce the spherical coordinates \( (r, \theta, \phi) \) having the polar axis along the \( S_n \) direction. Then we can integrate equation (36) with respect to \( \theta \) and \( \phi \) to obtain
\[ \Theta_s(\ell) = \frac{4\pi}{\omega \Sigma_s} \int_0^\infty P(r_1) \sin(\omega \Sigma_s r_1) \, dr_1 \]
\[ = -\frac{4\pi}{\omega \Sigma_s} \frac{1}{\xi(\omega \Sigma_s)} \int_{-\infty}^{\infty} P(r_1) \cos(\omega \Sigma_s r_1) \, dr_1 \]
\[ = -\frac{2\pi}{\omega \Sigma_s} \xi(\omega \Sigma_s) \hat{P}(\omega \Sigma_s), \]

where \( \hat{P}(\omega \Sigma_s) \) is the 1-D spatial Fourier transform of the parameter variation \( P(r) \).

We now show two examples, a uniform sphere and a local Gaussian type spherical heterogeneity to demonstrate the general scattering characteristics.

A uniform sphere

For a uniform sphere, \( P(r) \) is a boxcar function. The resulting volume factor is

\[ \Theta_s(0) = \frac{4\pi a^3}{(\omega \Sigma_s a)^2} \left[ \sin \frac{\omega \Sigma_s a}{\omega \Sigma_s} - \cos \frac{\omega \Sigma_s a}{\omega \Sigma_s} \right], \]

where \( a \) is the radius of the sphere. Note that

\[ \frac{\sin (\omega \Sigma_s a)}{\omega \Sigma_s a} - \cos \frac{\omega \Sigma_s a}{\omega \Sigma_s} \approx \frac{1}{3} (\omega \Sigma_s a)^2, \]

\[ \Theta_s(0) \approx V, \text{ when } \omega \Sigma_s a \ll 1. \]

In Rayleigh scattering, the phase differences between the volume elements are neglected, so the volume factor is equal to the real volume. For Mie scattering, the volume factor is generally smaller than the volume because of the interference between the fields from different volume elements. Figures 10–12 and 13–15 give the volume factors for \( S \)-wave scattering \( \Theta_S = \Theta_2 \) and for wave converting \( \Theta_C = \Theta_1 = \Theta_3 \), respectively. In plotting, we normalized the values with the volume of the sphere. It can be seen that for \( P-P \) or \( S-S \) scattering the volume factor \( \Theta \) always has the main lobe in the forward scattering direction, since all the scattered waves from different volume elements are always in-phase along the incident direction. When the volume becomes larger, small back lobes and side lobes start to appear (Figures 11 and 12), but most of the scattered energy for the common-mode scattering (\( P-P \) or \( S-S \)) will be contained in the main forward lobe. However, because the scattered waves have different velocities from the incident waves, destructive interference also occurs in the forward direction. Therefore the volume factor \( \Theta_C \) along the incident direction will reduce its value and will become oscillatory with increasing frequency. The scattering pattern then becomes more complex, and the converted energy diverges to all directions (e.g., Figure 15).

The general scattering pattern is a combination of \( E(\ell) \) and \( \Theta(0) \). Figures 16 through 26 show examples of scattering patterns. Note that for \( P-P \) or \( S-S \) scattering the main lobe becomes sharper and much bigger than the side lobes when frequency goes higher (Figure 17 and 22), while the converted waves and cross-coupled waves (\( \Theta_C \)) diverge into many small lobes in all directions and become much smaller compared with the main lobe (Figures 17, 22, and 26). Also remarkable is the
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**Fig. 14.** $\Theta^c$, when $(\alpha/\beta_0)\lambda t = 2\pi$.

**Fig. 15.** $\Theta^c$, when $(\alpha/\beta_0)\lambda t = 10$.

**Fig. 16.** Scattering patterns of a uniform sphere for $P$-wave incidence for different frequencies. When $\delta \lambda/\lambda_0 = \delta \mu/\mu_0 = 2(\delta \rho/\rho_0)$ and $\lambda_0 = \mu_0$. The upper half is of $P$-$P$ scattering, the lower half is of $P$-$S$ scattering. All the patterns are axially symmetric about the $x$-axis.

**Fig. 17.** Same as Figure 16 for $(\alpha/\alpha_0)\lambda t = 10$.

**Fig. 18.** Same as Figure 16, but for $\delta \lambda/\lambda_0 = \delta \mu/\mu_0 = \delta \rho/\rho_0$, i.e., the impedance type scattering.

**Fig. 19.** Same as Figure 18, when $(\alpha/\alpha_0)\lambda t = 10$. 
FIG. 20. Same as Figure 16, but for $\delta \mu /\mu_0 = -\delta \rho /\rho_0$, i.e., the velocity type scattering.

FIG. 21. Scattering patterns of a uniform sphere in the $x$-$y$ plane for $S$-wave incidence for different frequencies when $\delta \mu /\mu_0 = 2(\delta \rho /\rho_0)$. The upper half is of $S$-$S$ scattering, the lower half is of $S$-$P$ scattering. Note that there exists only the meridian components of scattered $S$-waves.

FIG. 22. Same as Figure 21 for $(\omega/\beta_0)\alpha = 10$.

FIG. 23. Same as Figure 21 but for $\delta \mu /\mu_0 = \delta \rho /\rho_0$, i.e., the impedance type scattering.

FIG. 24. Same as Figure 23 for $(\omega/\beta_0)\alpha = 10$.

FIG. 25. Same as Figure 21 but for $\delta \mu /\mu_0 = -\delta \rho /\rho_0$, i.e., the velocity type scattering.
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Fig. 26. Scattering patterns of scattered latitudinal component of a uniform sphere for S-wave incidence for different frequencies. The plane has a 45 degree angle to the x-y plane with a common x-axis.

impedance-type scattering which does not have a main lobe in the forward direction and has features similar to the converted or cross-coupled waves (Figures 18, 19, 23 and 24).

Since there is no latitudinal component of scattered S-waves in the x-y plane [see equation (22)], we took a plane with the common x-axis having 45 degree angle with the x-y plane (Figure 26) to present the changes of the scattering pattern with increase of frequency. Figure 26 gives the patterns in that plane for three frequencies. Note that there will be four small lobes of latitudinal component bending close to the x-axis when the wavelength becomes shorter, though the x-axis is a node. These four small lobes give rise to the scattered cross-component (here the z-component) in the nearly forward direction, which might be very useful in detecting the existence of scattering inclusions (Turpening, 1984).

A Gaussian spherical heterogeneity

Suppose the parameter variation of a spherical heterogeneity has a Gaussian shape, i.e.,

\[ P(r) = \exp \left( \frac{r^2}{a_o^2} \right) \]

where \( a_o \) is the characteristic length of the heterogeneity. From equation (37) we have

\[ \Theta_n = (\sqrt{\pi a_o}) \exp \left[ -(\alpha a_o a_o)^{-1} \right] \]

In Figure 27, \( \Theta^p \) are shown on the upper half-plane and \( \Theta^s \) on the lower half-plane. Compared with the case of a uniform sphere, the patterns for high-frequencies are much simpler. Because of the smoothness of the parameter variation, small side lobes do not appear in the high-frequency range. The scattered energy gradually concentrates to the forward lobe and becomes narrower and narrower without splitting when frequency goes higher and higher. Therefore the total scattering patterns also become simpler for high frequencies. Figures 28 and 29 show the patterns of impedance type and velocity type for P-wave incidence. Figure 30 shows the patterns for S-wave incidence only for \( (\omega / \beta_0) a_o = 10 \) to compare with the case of uniform sphere of Figure 24. The patterns for cross-coupled component \( \Theta^s_{us} \) are shown in Figure 31 and are presented in the same manner as in Figure 26. The four lobes gradually bend toward the forward direction without splitting when frequency increases.

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