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‘Single-sided’ autofocusing of sound in layered materials

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Abstract

A new technique—‘single-sided’ autofocusing—is proposed and its possible use for inspecting layered materials is surveyed. Based on linear acoustics and the recently developed mathematical theory of focusing, single-sided autofocusing answers the question, ‘Given single-sided access, how does one focus sound to a point inside a one-dimensional layered medium at a specified time—given that the velocity profile is unknown?’

1. Introduction

Ultrasonic nondestructive evaluation detects and characterizes undesirable features in structures and materials: for example, delaminations in composite material or cracks in welds. Focusing is perhaps the most common way of localizing such flaws and distinguishing them from the desired properties of the material or structure. Practical considerations often ensure that access to only one side of a structure is available. This often limits and complicates the focusing of ultrasound inside the sample. These limitations on focusing are particularly evident in the following simplified situation. Suppose one has a one-dimensional (1D) layered material where the sound velocity is constant for \(x < 0\) but then varies in an unknown way for \(x > 0\). Furthermore, suppose the inspection is carried out at normal incidence using plane-wave pulses. Finally, suppose the inspector has access to only the left half-space (single-sided access) but can prepare any desired normally plane-wave incident pulse and measure any resulting normally incident plane-wave reflected pulse. Then the problem is: ‘How can the inspector find an incident pulse such that the ultrasonic field focuses to a point inside the sample at some specified time?’

The ability to focus sound at a certain time or at a certain place has wide use. For example, lithotripsy, using ultrasonic shock waves to pulverize kidney stones, relieves a great deal of human suffering. Two generic problems arise in focusing. First, the incident field can become distorted due to variations in the velocity of sound and thus degrade the degree of focusing.
Second, the target may move and the waves may focus in the wrong place. Fink and his collaborators [1–6] solved both of these problems by developing a real-time system—based on the time-reversal invariance of the wave equation—to compensate for both phase aberration and target motion. Their insight has led to significantly improved ultrasonic imaging by linear acoustic arrays.

This paper is one of a series that grew out of an interesting observation of Prada and Fink et al [7–9], who discovered a form of autofocusing for three-dimensional (3D) acoustics. They showed that iteratively retransmitting an array’s own time-reversed signal causes it to focus on the strongest scatterer in the field of view. This important result is nonetheless limited in two ways. First, it does not measure the unknown sound velocity and consequently the focal position is unknown. Second, it only focuses on the strongest scatterer in the field of view (or the several strongest). Thus, one can, in principle, ‘see’ in small regions about these strong scatterers. The challenge is to focus the beam wherever one pleases and thus ‘see’ everywhere in the scatterer. Aktosun and Rose [10] met this challenge in a theorem–proof paper that contained the mathematical theory of single-sided focusing for the 1D variable velocity Helmholtz equation (VVHE). They show that one can focus to any point in 1D by constructing an incident pulse that consists of a delta function wavefront followed by the time-reversed solution of Marchenko’s equation [10, 11].

Here, I propose an automatic iterative procedure for 1D single-sided focusing at a specified time and suggest that this new procedure may be useful for characterizing layered materials. The new procedure, to be called ‘single-sided autofocusing’, iteratively updates the incident pulse based on the previous reflected pulse. More precisely, the \( n \)th incident pulse is updated by setting it equal to a sharp unipolar pulse (in the limit a delta function) minus the time-reversed and truncated \((n-1)\)th reflected pulse. Single-sided autofocusing requires only measured inputs and outputs. No equations need be solved.

This paper outlines one possible engineering application that follows from the mathematical theory of 1D single-sided focusing. It is written in the language of ultrasonic NDE (incident pulses, reflected pulses, impulse response function, etc) to improve the chances that it might result in a usable instrument. The more mathematically inclined may, in addition, want to consult [10].

This paper’s structure is as follows. The proposed autofocusing technique is described in section 2. Numerical simulations are used to test the proposed autofocusing technique in section 3. Section 4 shows how autofocusing is connected with previous work on single-sided focusing and inverse scattering. Section 5 summarizes this work and discusses the implications of autofocusing for the characterization of flaws in layered structures.

2. Single-sided focusing for the 1D variable velocity Helmholtz equation

The single-sided focusing problem is described and a step-by-step prescription is given for autofocusing. An ultrasonic inspection typically involves transmitting pulses into a sample and measuring the resulting reflected pulses, which are determined by the ultrasound transmitted into the sample and the sample’s velocity profile. The single-sided focusing problem is solved, by definition, when one finds an incident pulse such that, at some specified time, the ultrasound inside the sample is sharply peaked at some point—in the limit this peak is a delta function. Figure 1(a) shows the solution for one particular velocity profile.

The 1D VVHE is used to model wave propagation

\[
\frac{\partial^2 u(t, x)}{\partial x^2} - \frac{1}{c^2(x)} \frac{\partial^2 u(t, x)}{\partial t^2} = 0.
\]
Here, \( t \) and \( x \) denote the time and 1D spatial coordinate. The sound velocity, \( c(x) \), is assumed to be a real-valued and otherwise a smooth piecewise continuous positive function of \( x \). The sound velocity varies, possibly strongly, on the right half-line but is a constant \( c_0 \) on the experimentally accessible left half-line. The autofocusing procedure focuses the pulse at an instant that is conventionally and without loss of generality chosen to be zero. Furthermore, the incident pulse is chosen to have a delta function wavefront that crosses the origin of coordinates \( x = 0 \) at time \( t = -t_f \). Thus, the focal point \( x_f \) lies to the right of \( x = 0 \) (in the region where the velocity may be strongly varying) and corresponds to the distance travelled by the delta function during the interval \( t_f \).

The main result of this paper, a step-by-step prescription for autofocusing, now follows. Since the process is iterative, one describes the \( n \)th iteration.
Step 1

Interrogate the sample with the \( n \)th incident pulse, \( \phi_{\text{in}}(t - x/c_0; t_f) \), that commences with a sharp right-going unipolar pulse that crosses the origin at \( t = -t_f \) and is followed by any smooth trailing wave, a ‘tail’. See figure 1(a) for an example incident pulse. In the limit, the sharp unipolar pulse becomes the delta function \( \delta(t + t_f - x/c_0) \).

Step 2

Record the reflected left-going pulse on the left half-line to obtain \( \phi_{\text{out}}^{(n)}(t + x/c_0; t_f) \).

Step 3

Evaluate the result at \( x = 0 \) and truncate for \( t > t_f \): \( \phi_{\text{out}}^{(n)}(t; t_f)\Theta(t_f - t) \).

Step 4

Time reverse the result, \( t \rightarrow -t \), to obtain \( \phi_{\text{out}}^{(n)}(-t; t_f)\Theta(t_f + t) \).

Step 5

Subtract this result from \( \delta(t + t_f) \) to obtain \( \delta(t + t_f) - \phi_{\text{out}}^{(n)}(-t; t_f)\Theta(t_f + t) \).

Step 6

Let \( t \rightarrow t - x/c_0 \). The result is the new right-going incident pulse

\[
\phi_{\text{in}}^{(n+1)}(t - x/c_0; t_f) = \delta(t - x/c_0 + t_f) - \phi_{\text{out}}^{(n)}(-t + x/c_0; t_f)\Theta(t - x/c_0 + t_f); \tag{2}
\]

Step 7

Iterate until convergence.

Here, \( \Theta(x) \) equals one if \( x \geq 0 \) and zero otherwise.

The rest of the paper is a commentary on the prescription just given. The next part illustrates the step-by-step prescription for autofocusing; the following part connects it with known theory and the final part speculates on its possible uses for characterizing layered materials. How does single-sided focusing differ from ordinary focusing in pulsed systems? The following simple example, a step-function velocity profile, gives a good idea (see figure 2(a)). The velocity is \( c_0 \) to the left and \( c_1 \) to the right of the step at \( x_s \). By inspection, the analytic solution to this single-sided focusing problem for focusing after an interval \( t_f \) is the following incident pulse, shown in figure 2(b):

\[
\phi_{\text{in}}^f(t - x/c_0; t_f) = \delta(t - x/c_0 + t_f) - R\delta(t - x/c_0 + 2x_s/c_0 - t_f)). \tag{3}
\]

Here, \( R \) and \( T \) denote the reflection and transmission coefficients for the step. The total field generated by this incident pulse can have up to four transmitted and reflected delta functions. However, things are special at the focal time \( t = 0 \) because the second delta function has been chosen so that it exactly cancels the delta function reflection of the leading delta function \( \delta(t - x/c_0 + t_f) \)—see figure 2(c). Consequently, at \( t = 0 \) the total field consists of the single delta function

\[
\phi_{\text{out}}^f(t = 0, x; t_f) = T\delta(x_f/c_1 - x/c_1). \tag{4}
\]
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Thus, the sound field focuses to a delta function at $t = 0$. The focus is at a point $x_f$ that corresponds to the distance travelled by the wavefront after it crosses $x = 0$.

Next, the autofocusing prescription is shown to solve, in a systematic way, the problem of focusing in a step-function velocity profile.

**Step 1**

Send in an incident pulse $\delta(t - x/c_0 + t_f)$.

**Step 2**

Record the reflected pulse $R\delta(t + x/c_0 - 2x_s/c_0 + t_f)$.

**Step 3**

Evaluate the result at $x = 0$ and truncate everything that comes back after $t_f$ to obtain $R\delta(t - 2x_s/c_0 + t_f)$.

**Step 4**

Time reverse the result ($t \rightarrow -t$) to find $R\delta(t + 2x_s/c_0 - t_f)$.

**Step 5**

Subtract the result from the leading delta function to find $\delta(t + t_f) - R\delta(t + 2x_s/c_0 - t_f)$. 

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**Figure 2.**

(a) The transmitted and reflected pulses (vertical arrows) due to the incident delta function $\delta(t - x_f/c_0)$. The horizontal lines labelled $c_0$ and $c_1$ show the velocity profile. Finally, the horizontal arrows show the direction of propagation. (b) The incident field that leads to focusing at $x_f$, as shown in the next panel. It consists of two delta functions: the first with strength one and the second with strength $-R$. (c) The field focused at $x_f$, as shown by the solid vertical arrow. The down-going dashed arrow shows the reflected pulse, which is being cancelled by the second pulse of the incident field.
Step 6

Replace \(t\) by \(t - x/c_0\) to obtain the new incident pulse
\[
\phi^{(2)}_{in}(t - x/c_0; t_f) = \delta(t - x/c_0 + t_f) - R \delta(t - x/c_0 + 2x/c_0 - t_f). \tag{5}
\]

Equation (5) is equivalent to equation (3), the exact solution for the step-function profile, which was previously obtained by inspection. The solution followed from a single iteration of the autofocusing prescription that was described above. More iterations leave equation (5) unchanged since step 3 of the prescription truncates reflections from the second delta function.

The solution for the step-function velocity profile illustrates several important features and limitations of single-sided autofocusing. Autofocusing guarantees that the incident pulse focuses after a time interval \(t_f\). However, it does not directly determine the focal position \(x_f\) and the focal strength since the velocity on the right half-line remains unknown in general. Furthermore, note that although the total sound field is concentrated to a point, the energy is not—since, although the two cancelling delta functions do not contribute to the sound field at \(t = 0\), they do contribute to the kinetic energy. Section 5 shows how to deal with these limitations.

Note that the autofocusing procedure truncates the received pulse before retransmitting it. The essential nature of the truncation step can be seen in the example just discussed. Without truncation the procedure would lead to an infinite regress. Truncation arises because the incident and reflected fields are antisymmetric in time over a certain time interval (see section 4 and the appendix). Truncation ensures that only this time interval is considered when the \((n + 1)\)th incident field is inferred from the \(n\)th reflected field.

Autofocusing will need to converge relatively rapidly if it is to be widely useful. In the next section, I report numerical simulations for a number of different velocity profiles and note that for these particular examples convergence was obtained in 20 iterations, often many less. In addition, I note that autofocusing is the engineering equivalent of iterating Marchenko’s integral equation, the equation that governs much of 1D inverse scattering theory. It is known that Marchenko’s equation may be solved by iteration \cite{12} for a broad class of potentials, or equivalently velocity profiles. Consequently, it is reasonable to conjecture that the autofocusing procedure may also converge for the same broad class.

A formal connection between autofocusing and Marchenko’s equation will be shown below. First, define the impulse response function, \(R(t)\), to be the reflected field, measured at \(x = 0\), due to the incident delta function \(\delta(t - x/c_0)\). Also, note that for any incident field, \(\psi_{in}(t - x)\), the reflected field measured at \(x = 0\), \(\psi_{out}(t)\), is given by a convolution of \(\psi_{in}\) and \(R\), i.e.
\[
\psi_{out}(t) = \int_{-\infty}^{\infty} dt' R(t - t')\psi_{in}(t'). \tag{6}
\]

Next, start the autofocusing procedure by sending in an initial incident pulse \(\delta(t + t_f - x/c_0)\) and iterate according to the autofocusing prescription to obtain the following series:
\[
\phi_{out}(t; t_f) = R(t + t_f) - \int_{-\infty}^{t_f} dt' R(t + t')R(t' + t_f)
+ \int_{-\infty}^{t_f} dt' \int_{-\infty}^{t'} dt'' R(t + t')R(t' + t'')R(t'' + t_f) + \cdots. \tag{7}
\]

The general term of order \(N\) consists of \(N\) impulse response functions integrated together in the same fashion. Finally, sum the series to obtain the integral equation
\[
\phi_{out}(t; t_f) = R(t + t_f) - \int_{-\infty}^{t_f} dt' R(t + t')\phi_{out}(t'; t_f). \tag{8}
\]
Upon defining \( w(t, t_f) \equiv -\phi_{\text{out}}^f(t; t_f) \), equation (8) can be rewritten as
\[
0 = R(t + t_f) + w(t, t_f) + \int_{-\infty}^{t_f} dt' R(t + t') w(t', t_f).
\] (9)

This is, as advertised, Marchenko’s integral equation. I have just shown that the solution to Marchenko’s integral equation can be obtained, in principle, by autofocusing a piece of hardware and then measuring the reflected pulse.

3. Numerical simulation of autofocusing

Autofocusing was tested by numerical simulation for several different velocity profiles. I coded the autofocusing procedure outlined in the last section as a double-precision FORTRAN program. The core of the program is a finite-difference subroutine that simulates sound propagation through a spatially varying velocity profile. Time and space are discretized in 30,000 and 6000 equal intervals. A narrow Gaussian pulse, with a half-width of 0.10 and an integrated strength of one, was used to approximate the delta function. The code converged rapidly to a focus in all cases tested (20 or fewer iterations).

Figure 1(a) shows the incident and focused waves for a travel time \( t_f = 5 \). The velocity profile is given analytically by
\[
c(x) = 1.0 + e^{-|x-10|^2}.
\] (10)
The procedure converged to high accuracy after 15 iterations. Propagation through the velocity profile ‘filtered’ the incident pulse to obtain the sharply focused peak shown. Note that the focused peak is both higher and broader than the leading part of the incident pulse. Figure 1(b) shows both the incident field measured at \( \tau = -t_f \), and the ‘final’ field measured at \( \tau = t_f \). Note that the incident and final fields have opposite and equal values to the left of the incident wavefront. This follows immediately from the autofocusing prescription, which says that the \( n \)th incident pulse is just the delta function minus the \((n - 1)\)th truncated reflected pulse. Consequently, the antisymmetry noted above follows immediately when the autofocus procedure converges. This antisymmetry is the key to autofocusing and is the subject of section 4. Finally, figure 1(c) shows the field just before focusing, at focusing and just after the focusing.

Figure 3 shows focusing for travel time \( t_f = 5 \) for the strongly varying velocity profile
\[
c(x) = 1 + 3e^{-|x-9|^4}.
\] (11)

Note that the maximum velocity is four times the background velocity. This figure shows the incident, the focused and the final waves. For this example, autofocusing converged to high accuracy after 20 iterations. As can be seen, the incident pulse has a complicated and spatially extended form. Propagation through the potential filters out these complications and compresses the incident pulse into a simple Gaussian.

Autofocusing converges extremely rapidly for the examples tested. Figure 4 illustrates the rapidity of convergence for the strongly varying velocity profile
\[
c(x) = 1 - 0.6666666e^{-|x-11|^4}.
\] (12)

Figure 4(a) shows the field due to an incident delta function: i.e. no correction for focusing. Figure 4(b) shows that focusing is nearly achieved after the first iteration. Figure 4(c) shows that focusing to graphical accuracy is found with the second iteration.
Figure 3. The incident, the final and the focused waves for propagation through the velocity profile shown. Notice that the incident and final pulses have opposite and equal values for positions to the left of the incident wavefront at 5.0.

Figure 5 shows both the incident pulse and the focused pulse for a discontinuous velocity profile that has the form

\[ c(x) = \begin{cases} 
1.0 & 0 < x < 8 \\
1.5 & 8 < x < 9 \\
0.5 & 9 < x < 10.
\end{cases} \] (13)

Note that the velocity profile differs at plus and minus infinity. The incident pulse consists of the incident delta function plus three trailing peaks. They are filtered out by propagation and the wave is focused at \( t_f = 4.0 \).

4. Relation of autofocusing to the mathematical theory of focusing

The following antisymmetry is the key to autofocusing. The incident and reflected fields are related by

\[ \phi_{in}^f(t; t_f) = -\phi_{out}^f(-t; t_f), \] (14)
during the interval \(-t_f < t < t_f\). That is, focusing causes the incident and reflected sound pulses to be antisymmetric pairs. See figures 1(a) and (3). Consequently, any measured reflected pulse suggests in turn an incident field—via equation (14). This is the key that allows one to iterate the focusing process.

Reference [10] contains a rigorous proof of the antisymmetry that applies \textit{inter alia} to the VVHE. On the other hand, [11] contains a more physically transparent derivation of antisymmetry—but for the plasma wave equation. The appendix extends the arguments of [11] to the VVHE. Reference [10] also proves that, for a broad class of continuous velocity profiles, the incident field that focuses at time \( t_f \) consists of an incident delta function followed by the time-reversed solution of Marchenko’s equation

\[ \phi_{in}^f(t; t_f) = \delta(t + t_f) + w(-t; t_f). \] (15)
5. Discussion of material’s characterization and summary

Two possible uses of single-sided autofocusing come immediately to mind: communication and inspection. In this section, the possible use of autofocusing for a material’s characterization is suggested and some of its possible limitations are remarked on. Single-sided focusing has one major virtue; it focuses the sound field. Its major vice is that it does not focus the energy and this complicates its use for characterizing materials. To see this point clearly, consider the step-function velocity profile discussed in section 2 and illustrated in figure 2. At $t = 0$ the sound concentrates into a single pulse. However, from another point of view, there are still three pulses propagating through the sample; it is just that two cancel at $t = 0$. Given this general feature of single-sided focusing, it is very reasonable to ask, ‘Can the velocity profile be measured by single-sided autofocusing?’ and ‘Does single-sided focusing add anything to our current understanding of 1D inverse scattering methods, which find the velocity profile from the impulse response function $R(t)$?’
The answer to the first question is ‘Yes, the velocity profile can be determined if one has autofocused the system and thus found the focusing incident field’. As shown in section 2, autofocusing measures the solution to Marchenko’s equation—and thus replaces the first and hardest step of 1D inverse scattering theory. The velocity profile can then be found in the standard way by solving a standard ordinary differential equation (ODE) [10–13]. The second question, concerning advances with respect to 1D inverse scattering theory, is unanswered. Autofocusing—because of its proposed measurement capability—avoids the solution of Marchenko’s equation altogether. Furthermore, the structure of autofocusing theory suggests that the velocity profile may be recoverable from the solution of an integral equation, rather than an ODE. Autofocusing may afford an improvement over traditional inverse scattering theory on both these counts. However, such questions remain open and are reserved for future research.

The autofocusing inverse scattering problem can be formalized as follows. Suppose a piece of hardware can autofocus for any desired focusing time $t_f$. Then, determine the velocity profile using the output of this device—i.e. $\phi_{in}^f(t; t_f)$, the incident field required for focusing for all focusing times $t_f$. The statement of the problem may be further refined for continuous and smooth velocity profiles. Consider the focusing incident pulse shown in figure 3. Define $\beta(t_f)$ to be the value of this pulse just to the left of the sharp unipolar leading edge (in the limit that the leading edge becomes a delta function). Mathematically, $\beta(t_f) \equiv \phi_{in}^f(t \to -t_f + \epsilon; t_f)$, where $\epsilon$ is positive and infinitesimal. The inverse problem is now to find the velocity profile given $\beta(t_f)$ for all $t_f$.

The approaches that I am aware of for characterizing materials via autofocusing require a certain degree of smoothness for the velocity profile. Scattering depends on the relative size of the wavelength and the length scale over which the potential varies. Waves of sufficiently high frequency are not scattered back but simply ‘guide’ through the velocity profile. This has two consequences. First, propagation leaves a delta function unchanged except for its velocity and strength—this follows from 1D geometrical optics transformed to the time domain. Second, backscattering is weak at high frequencies. This weakness complicates all methods for characterizing changes in the velocity from backscatter and will be a limitation for methods based on autofocusing.

In summary, a step-by-step prescription has been given for single-sided autofocusing. The ability of this procedure to focus the total field was tested by numerical simulation. The mathematical theories of focusing and inverse scattering were related to the new procedure. Finally, I introduced the possible use of autofocusing to determine the velocity profile.
Appendix

This appendix derives equation (14), which shows that the incident field can be determined from the reflected fields by antisymmetry. The derivation follows that given in [11] for the plasma wave equation. It first shows that the focused total field at \( x = 0, \phi_{\text{tot}}^f(t, x = 0; t_f) \), is antisymmetric with respect to time over the time interval \( -t_f < t < t_f \). Then the derivation shows that this implies equation (14).

The VVHE is a second order in time differential equation. The total field at \( x = 0, \phi_{\text{tot}}^f(t, x = 0; t_f) \), is determined by the wave equation and the initial value data, which are the wave field and its time derivative for all \( x \) at \( t = 0 \). However, knowing the initial value data for all \( x \) is infinitely more information than is needed to determine \( \phi_{\text{tot}}^f(t, x = 0; t_f) \) during the interval \( -t_f < t < t_f \). Remember that the velocity is bounded in this problem. Hence, the only \( t = 0 \) values of the data that can contribute to the field at \( x = 0 \) are those at positions such that their travel times are less than \( t_f \). That is, we do not need to know the initial value data for points such that \( x < x_f \). It suffices to know \( \phi_{\text{tot}}^f(t, x; t_f)|_{x=0} \) and its time derivative, \( \partial \phi_{\text{tot}}^f(t, x; t_f)/\partial t|_{t=0} \), both on the spatial interval \( -\infty < x < x_f \).

The antisymmetry of \( \phi_{\text{tot}}^f(t, x = 0; t_f) \) arises because at \( t = 0 \) the total field is zero on \( -\infty < x < x_f \) by the definition of focusing. Consequently, following the arguments of the preceding paragraph, we see that \( \phi_{\text{tot}}^f(t, x = 0; t_f) \), for times \( -t_f < t < t_f \), depends only on the time-derivative of the total field at \( t = 0 \), i.e. \( \partial \phi_{\text{tot}}^f(t, x; t_f)/\partial t|_{t=0} \).

All the elements are now in place to show the antisymmetric nature of the total field. One proceeds by noting that if the data are evolved forward in time from \( t = 0 \), one finds \( \phi_{\text{tot}}^f(t, x = 0; t_f) \). If the same data are propagated backwards in time, one obtains \( -\phi_{\text{tot}}^f(t, x = 0; t_f) \). This latter form is obtained by

(a) time-reversing the data \((t \to -t)\),
(b) propagating this new problem to its future, and
(c) time-reversing the solution \((-t \to t)\) to obtain the original problem.

The overall minus sign arises because the data, \( \partial \phi_{\text{tot}}^f(t, x; t_f)/\partial t|_{t=0} \), change sign when \( t \to -t \). Upon completing steps (a)–(c), the desired antisymmetric property of the total field is found:

\[
\phi_{\text{tot}}^f(t, x = 0; t_f) = -\phi_{\text{tot}}^f(-t, x = 0; t_f)
\]  
(A.1)

for \( -t_f < t < t_f \).

The antisymmetry of the total field at \( x = 0 \) implies that the incident and reflected fields are also related by antisymmetry. The total field on the uniform left half-line is just the incident plus the reflected field. Hence, for \( x = 0 \), equation (A.1) can be rewritten as

\[
\phi_{\text{in}}^f(t; t_f) + \phi_{\text{out}}^f(t; t_f) = -\phi_{\text{in}}^f(-t; t_f) - \phi_{\text{out}}^f(-t; t_f)
\]  
(A.2)

for \( -t_f < t < t_f \). Finally, equate the left-going waves on both sides of (A.1) to find

\[
\phi_{\text{out}}^f(t; t_f) = -\phi_{\text{in}}^f(-t; t_f).
\]  
(A.3)

for \( -t_f < t < t_f \).
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