Earthquake Simulation by Restricted Random Walks

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Abstract. This article simulates earthquake slip distributions as restricted random walks. Random walks offer several unifying insights into earthquake behaviors that physically-based simulations do not. With properly tailored variables, random walks generate observed power law rates of earthquake number versus earthquake magnitude (the Gutenberg-Richter relation). Curiously b-value, the slope of this distribution, not only fixes the ratio of small to law events but it also dictates diverse earthquake scaling laws such as mean slip versus fault length and moment versus mean slip. Moreover, b-value determines the overall shape and roughness of earthquake ruptures. For example, mean random walk quakes with b=-1/2 have elliptical slip distributions characteristic of a uniform stress drop on a crack. Random walk earthquake simulators, tuned by comparison with field data, provide improved bases for statistical inference of earthquake behavior and hazard.

1. Introduction.

Imagine hiking the trace of a new earthquake rupture and measuring surface slip offsets at many points along the fault strike (see Figure 1, for examples). This mapping exercise may be akin to a random walk in the sense that the measured offset at position \( n \) gives scant hint as to whether the offset at position \( n+1 \) will be larger or smaller.

Consider then, earthquake slip functions \( u(n\Delta L) \) sampled at regular intervals \( \Delta L \) as simulated in

\[
u(n\Delta L) = u_0 + \Delta u \sum_{i=1}^{n} \chi_i(0, \sigma_i)
\]  

Here, \( \Delta u \) is some small slip value and the \( \chi(\nu, \sigma) \) are unitless random variables with mean \( \nu \) and variance \( \sigma^2 \). Random walk "ruptures" (1) start with \( u_0 = \Delta u \) and run a distance \( L=N\Delta L \) to where \( u(N\Delta L) \leq 0 \)

Figure 1. Measured dextral displacement along the surface fault trace of the Sakarya Segment, M7.4 Ismit, Turkey earthquake of August 17, 1999 (Barka et al., 2002) and the M7.1 Duzce, Turkey earthquake of November 12, 1999 (Akyuz et al., 2002).

Figure 2. Sketch of an earthquake rupture simulated by a restricted random walk. The rupture starts at \( l=0 \) with slip \( \Delta u \) and runs down strike taking random fluctuations in slip until \( u(L) \) first falls to zero or below.
Random walks like these, that terminate at a barrier are labeled restricted or absorbing. The extent to which simulation (1) might represent earthquakes hinges upon the selection of the random variables. In a leap of faith, suppose that the synthetic ruptures could indeed be made to reproduce many earthquake behaviors. If so, then random walks would, at the least, provide simple statistical bases for testing rupture hypotheses. Strong leapers might even imagine random walks leading to insights into earthquake behavior that physics-based rupture simulations obscure in curtains of complexity.

2. Walk Statistics.

Unrestricted Walks Reaching Length L. Envision initially, unrestricted random walks with \( u_0 = 0 \) that run length \( L \) as shown in Figure 3. Let \( P_{u(L)}(u) \) be the probability of \( u(L) \) falling between \( u \) and \( u + \Delta u \). If the random variables \( \chi_n \) are well enough behaved such that the central limit theorem holds, then for large \( L \) \( P_{u(L)}(u) \) would be normally distributed with zero mean and variance \( \sigma^2(L) \)

\[
P_{u(L)}(u) = \frac{\Delta u}{2} \frac{\partial}{\partial u} \text{erf}\left(\frac{u}{2\Delta \sigma(L)}\right) \quad (2)
\]

The \( \text{erf}(x) \) here is the integrated Gaussian or error function (see Appendix A). The variance of unrestricted walk heights at \( L \) is defined by

\[
\sigma^2(L) = \Delta u \int_{-\infty}^{\infty} u^2 P_{u(L)}(u) du
\]

\[
= \Delta L \int_0^L \Delta \sigma^2(l) dl
\]

where \( \Delta \sigma \) is the variance increment along the path from 0 to \( L \). For independent \( \chi_n \) in (1)

\[
\Delta \sigma^2(n \Delta L < l < (n + 1) \Delta L) = \Delta u^2 \sigma_n^2
\]

and

\[
\sigma^2(L) = \Delta u^2 \sum_{n=1}^{N=L/\Delta L} \sigma_n^2 \quad (4)
\]

Restricted Walks Reaching Length L. When the
walk is restricted to positive offsets only and $u_0 = \Delta u$, then the probability of $u(L)$ falling between $u$ and $u + \Delta u$ becomes

$$P_{u(L)}(u, \Delta u) = P_{u(L)}(u - \Delta u) - P_{u(L)}(u + \Delta u)$$

and

$$= \frac{\Delta u}{2} \frac{\partial}{\partial u} \left[ \text{erf} \left( \frac{u + \Delta u}{\sqrt{2} \sigma(L)} \right) - \text{erf} \left( \frac{u - \Delta u}{\sqrt{2} \sigma(L)} \right) \right]$$

(5)

I sketch equation (5) toward the right in Figure 4. Integrating $P_{u(L)}(u, \Delta u)$ over all values $u$, gives the probability that the restricted walk runs a distance $L$ or greater

$$P_>(\Delta u, L) = \int_0^\infty P_{u(L)}(u, \Delta u) du = \text{erf} \left( \frac{\Delta u}{\sqrt{2} \sigma(L)} \right) \approx \sqrt{\frac{2}{\pi}} \frac{\Delta u}{\sigma(L)}$$

(6)

The probability that the restricted walk actually ends between $L$ and $L + \Delta L$

$$P(\Delta u, L) = \Delta L \frac{\partial P_>(\Delta u, L)}{\partial L}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\Delta u \Delta L}{\sigma^2(L)} e^{-\frac{\Delta u^2}{2 \sigma^2(L)}} \frac{\partial \sigma(L)}{\partial L}$$

$$\approx \sqrt{\frac{2}{\pi}} \frac{\Delta u \Delta L}{\sigma^2(L)} \frac{\partial \sigma(L)}{\partial L}$$

(7)

is just the fraction that reach distance $L$ times a positive number less than one. In the restricted walk, the expected value and variance of the heights at $L$ are

$$\begin{align*}
E[u(L)] &= \Delta u \left[ \text{erf} \left( \frac{\Delta u}{\sqrt{2} \sigma(L)} \right) \right]^{-1} \approx \sqrt{\frac{\pi}{2}} \sigma(L) \\
\text{Var}[u(L)] &= \sigma^2(L) + \Delta u^2 + \sqrt{\frac{2}{\pi}} \sigma(L) e^{-\frac{\Delta u^2}{2 \sigma^2(L)}} \left[ E[u(L)] - E[u(L)] \right]^2 \\
&\approx (2 - \pi/2) \sigma^2(L) = 0.43 \sigma^2(L)
\end{align*}$$

(8a)

and

$$\begin{align*}
\text{Var}[u(L)] &= \sigma^2(L) + \Delta u^2 + \sqrt{\frac{2}{\pi}} \sigma(L) e^{-\frac{\Delta u^2}{2 \sigma^2(L)}} \left[ E[u(L)] - E[u(L)] \right]^2 \\
&\approx (2 - \pi/2) \sigma^2(L) = 0.43 \sigma^2(L)
\end{align*}$$

(8b)

The approximate versions of (6-8) hold for long ruptures where $\sigma(L) \gg \Delta u$. Restricted walks have a non-zero mean value with a spread in offsets about half as large as unrestricted walks of equal length.

Random walk theory (6) predicts an inverse relationship between the length-survivability of earthquake ruptures and the degree to which slip varies along strike. To my knowledge, physics-based models of earthquake rupture have not made such a prediction. In particular, random walk quakes with large step-to-step variations have less chance of making long ruptures than do quakes with small step-to-step variations. Bear in mind, that the variation in earthquake slip along strike, $\sigma^2(L)$ is field-measurable, so means exist to test and tune the simulator. Slip variation forms a common thread through all predictions below.

Restricted Walks Terminating Length $L$. Consider finally restricted random walks that terminate at $L$ (Figure 5). These walks
resemble earthquake ruptures of length $L$ and I label their offset along strike $u_L(l)$. Because the offset vanishes at both ends of the rupture, the variance $\sigma^2_L(l < L)$ of $u_L(l)$ increases like (4) initially, but then tapers to zero at $L$

$$\sigma^2_L(l) = \frac{\sigma^2(l)}{\sigma^2(L)}\left[\sigma^2(L) - \sigma^2(l)\right]$$

From (8a), the expected value of slip at distance $l$ in such ruptures would be

$$E[u_L(l)] \approx \sqrt{\frac{\pi}{2}} \sigma_L(l)$$

Thus, the mean slip of all earthquakes of length $L$ depends on along-strike slip variability like

$$\bar{U}(L) = \frac{\sqrt{\pi/2}}{L\sigma(L)} \times \left[\int_0^L \sigma(l)\left[\sigma^2(L) - \sigma^2(l)\right]^{1/2} dl\right]$$

Seismologists have particular interest in the seismic moment of earthquakes

$$M_o = \mu WL\bar{U}$$

where $\mu$ is the elastic rigidity of the crust, $\bar{U}$ is the mean slip in the event, and $W$ is the down dip (into the Earth) width of ruptures in the region of interest. [W is assumed constant here.] From (11) and (12), the mean moment in all quakes terminating at $L$ would be

$$\bar{M}_0(L) = \mu WL\bar{U}(L)$$

Once our simulation specifies slip variability $\sigma^2(L)$, then $\bar{M}_0(L)$ can be inverted to provide $\bar{L}(M_0)$, the mean rupture length needed to generate a given seismic moment. Back substitution of $\bar{L}(M_0)$ into (6) then provides the probability that simulated earthquakes exceed a given moment

$$P_{>}(\Delta u, M_0) \approx \frac{2}{\pi} \frac{\Delta u}{\sigma(\bar{L}(M_0))}$$

3. Application to earthquakes.

Predicted Rate Distribution. What does this theory really have to do with earthquakes? Suppose that I find a set of random variables $\chi_n$ such that standard deviation of slip along the unrestricted walk (4) grows like

$$\sigma(l) \approx \Delta u(l / \Delta L)^q$$

Evaluating integral (11) then, provides the mean slip in quakes of length $L$
\[
\bar{U}(L) = \Delta u \sqrt{\frac{\pi}{8}} \sqrt{\frac{\pi}{2q}} \frac{\Gamma(\frac{1+q}{2q})}{\Gamma(\frac{1+4q}{2q})} \left( \frac{L}{\Delta L} \right)^q
\]  
(16)

\[= \Delta u K(q) \left( \frac{L}{\Delta L} \right)^q\]

Placing (16) into (13) and inverting for the mean rupture length for quakes with moment \(M_0\), I find

\[
\bar{L}(M_0) = \Delta L \left( \frac{M_0}{\mu \Delta u W \Delta L K(q)} \right)^{\frac{1}{q+1}}
\]  
(17)

\[= \Delta L \left( \frac{M_0}{\Delta M_0 K(q)} \right)^{\frac{1}{q+1}}\]

and (15) gives

\[
\sigma(\bar{L}(M_0)) = \Delta u \left( \frac{M_0}{\Delta M_0 K(q)} \right)^{\frac{q}{q+1}}
\]  
(18)

with \(\Delta M_0 = \mu \Delta u W \Delta L\). From (14) the probability that a quake grows to moment \(M_0\) or greater is

\[
P_\geq(M_0) = \sqrt{\frac{2}{\pi}} [\Delta M_0 K(q)]^{q/(q+1)} M_0^{-q/(q+1)}
\]  
(19)

Finally, using the moment-magnitude relation \(M_w = (2/3) \log(M_0) - 9.05\), I write (19) as

\[
\log N_\geq(M_w) = \log N_\geq(M_w^{\text{min}})
\]

\[+ \left[ -\frac{3}{2} \frac{q}{1+q} \right] (M_w - M_w^{\text{min}})
\]  
(20)

Equation (20) specifies a power-law distribution in the rate of quakes of magnitude greater than \(M_w\). That is, random walk simulations (1) with variance (15) produce earthquakes that obey the Gutenberg Richter relation with

\[
b = -\frac{3}{2} \frac{q}{1+q} \quad \text{or} \quad q = \frac{-2b}{3+2b} \quad (21)
\]

Equation (21), the first clear tie of random walks with earthquake behaviors, offers initial justification for that leap in Section 1.

**Predicted Scaling Laws.** From (16) and (13) other earthquake scaling laws too can be written in terms of b-value

\[
\bar{U}(L) = \Delta u K(q) \left( \frac{L}{\Delta L} \right)^{-2b} = K(q) \sigma(L)
\]  
(22)

\[
\bar{M}_0(L) = \Delta M_0 K(q) \left( \frac{L}{\Delta L} \right)^3 = K(q) \bar{M}_0(L)
\]  
(23)

\[
\bar{U}(M_0) = \Delta u K(q) \left( \frac{M_0}{\Delta M_0} \right)^{3+2b} = K(q) \bar{U}(M_0)
\]  
(24)

Remarkably, b-value not only dictates the ratio of large to small earthquakes, but it also fixes the slope of many of their scaling laws. I know of no physics-based model of earthquake rupture that makes this link. In fact, random walk theory predicts that not only are slopes of the scaling laws interrelated but so are the leading constants. \((K(q)\) is given in Table 1.) Reproduction of observed scaling law slopes and their constants would be a strong endorsement of random walk simulations.
For earthquakes in bulk, a b-value of -1 is nearly universal. Random walk quakes with b=-1 scale like
\[ U \propto L^2 \; ; \; M_0 \propto L^3 \] and \[ U \propto M_0^{2/3} \] (25)

There is no reason however, that the simulator need reproduce bulk quake rates especially if it is intended to model events on a particular fault. Earthquakes restricted to the vicinity of a particular fault can have b-values considerably greater than -1 because isolated faults often produce many similar-sized events. Table 1 shows that seismic sources that vary in b-value by only a few tenths generate earthquakes that scale quite differently from (25). A dependence of scaling laws on the underlying b-value of the earthquake source could be a core confusion in quantifying earthquake behavior. Comparing rupture statistics from mixed populations of different b-valued sources would create a wide variability in observed earthquake scaling.

**Predicted Slip Function Features.** From (9), (10) and (15) the mean shape of earthquake slip functions in ruptures of length \( L \) is

\[ E[u_L(l)] = 2\bar{U}_{\text{peak}} \frac{L^q}{L^{2q}} \left( L^{2q} - l^{2q} \right)^{1/2} \] (26)

with \[ \bar{U}_{\text{peak}} = \Delta u \sqrt{\frac{\pi}{8}} \left( \frac{L}{\Delta L} \right)^q \] (27)

Amazingly, random walk theory says that b-value also controls the mean shape of earthquake slip functions. Figure 6 (left) plots (26) for b=-1/4 to b=-1. Note that the mean slip from b=-1/2 sources

<table>
<thead>
<tr>
<th>q</th>
<th>b</th>
<th>( U(L) )</th>
<th>( U(M) )</th>
<th>( M(L) )</th>
<th>( K(q) )</th>
<th>( \bar{U}_{\text{peak}} (L) )</th>
<th>( U_{\text{field peak}} (L) )</th>
</tr>
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<tr>
<td>2</td>
<td>-1</td>
<td>( \bar{u} \propto L^2 )</td>
<td>( \bar{u} \propto M_0^{2/3} )</td>
<td>( \bar{M}_0 \propto L^3 )</td>
<td>0.300</td>
<td>2.08</td>
<td>3.52</td>
</tr>
<tr>
<td>3/2</td>
<td>-0.9</td>
<td>( \bar{u} \propto L^{3/2} )</td>
<td>( \bar{u} \propto M_0^{0.6} )</td>
<td>( \bar{M}_0 \propto L^{5/2} )</td>
<td>0.351</td>
<td>1.78</td>
<td>3.00</td>
</tr>
<tr>
<td>1</td>
<td>-3/4</td>
<td>( \bar{u} \propto L )</td>
<td>( \bar{u} \propto M_0^{1/3} )</td>
<td>( \bar{M}_0 \propto L^2 )</td>
<td>0.417</td>
<td>1.50</td>
<td>2.52</td>
</tr>
<tr>
<td>1/2</td>
<td>-1/2</td>
<td>( \bar{u} \propto L^{1/3} )</td>
<td>( \bar{u} \propto M_0^{1/3} )</td>
<td>( \bar{M}_0 \propto L^{3/2} )</td>
<td>0.492</td>
<td>1.27</td>
<td>2.13</td>
</tr>
<tr>
<td>1/5</td>
<td>-1/4</td>
<td>( \bar{u} \propto L^{1/5} )</td>
<td>( \bar{u} \propto M_0^{1/6} )</td>
<td>( \bar{M}_0 \propto L^{6/5} )</td>
<td>0.477</td>
<td>1.31</td>
<td>2.20</td>
</tr>
</tbody>
</table>

Table 1. Earthquake Scaling Laws versus b-value as predicted from random walk theory.

![Figure 6](image) (Left) Mean shape of earthquake slip functions versus b-value of the source. (Center) 2-D Stress drops associated with the mean slip functions. (Right) Typical realizations of ruptures with the given b-value.
\[ E[u_L(l)] = \frac{2U_{\text{peak}}}{L} [l(L-l)]^{1/2} \] (28)

follows exactly the elliptical shape expected from a uniform stress drop of

\[ \Delta \tau = -\frac{\mu U_{\text{peak}}}{L} \] (29)

over a 2-D shear crack of length \( L \) (middle Figure 6). Equation (29) is another indication that fundamental relationships exist between random walks and earthquake physics. As b-value decreases to \( b = -1 \), the crack-like slip functions evolve to a dogtail/rainbow appearance (Ward, 1997). In these ruptures, stress drops over the concave down part of the rupture (the rainbow) while stress increases over the concave up part (the dogtail). Both dogtail/rainbow and elliptical slip functions are observed frequently in the field. Note too that ruptures from \( b < -1/2 \) sources are intrinsically skewed in walks (15). Skewness of slip functions is a field measurable quantity and this prediction too should be testable.

From (22), the peak mean slip to mean slip ratio

\[ \left( \frac{U_{\text{peak}}(L)}{U(L)} \right) = \sqrt{\frac{\pi}{8}} K^{-1}(q) \] (30)

also depends on b-value but it is independent of the length of rupture. Peak slip to mean slip ratio might be thought of as an unrecognized earthquake scaling law. Random walk theory says that slip functions from low b-value earthquake sources should be more "peaky" than those from higher b-value sources. As mentioned above, high b-value sources produce a larger proportion of characteristic earthquakes than do low b-value sources. Reduced slip variability for more characteristic sources makes sense. Peak mean slip to mean slip ratio (30) goes from 1.3 to 2.1 as b value decreases from -1/2 to -1. Equation (30) however must underestimate field-measured peak slip to mean slip ratios. Smooth shapes (26) represent the slip averaged over all quakes of length \( L \). Individual rupture events randomly perturb these shapes (see far right Figure 6) and will certainly peak out higher. From (8b) and (10) the variance in mean peak height is

\[ \text{Var}[\bar{U}_{\text{peak}}] = (2 - \pi / 2) \left( \frac{2}{\pi} \bar{U}_{\text{peak}}^2 \right) \]

Likely field-measured peak slip might exceed the mean value by 2 standard deviations

\[ U_{\text{peak}}^\text{field} \approx \left[ 1 + 2 \times (2 - \pi / 2) \sqrt{\frac{2}{\pi}} \right] U_{\text{peak}} = 1.68 U_{\text{peak}} \]

so a better, metric of slip variability might be

\[ \left( \frac{U_{\text{field}}(L)}{U(L)} \right) = 1.68 \times \sqrt{\frac{\pi}{8}} K^{-1}(q) \] (31)

Equation (31) goes from 2.2 to 3.5 as b value increases from -1/2 to -1. Predictions of slip function "peakyness" as measured by the ratio of peak slip to mean are field-testable.
We have seen now that random walk theory makes many specific predictions about the behaviors of earthquake ruptures. The b-value controls not only the scaling law constants but it also affects the "look and feel" of the slip distribution. I believe that correctly mimicking the look and feel of earthquake ruptures is critically important both in the tuning of the simulator and in assuring its practical use.

4. Design of the Walks- Numerical Results.

Basic Walk. The analytical results above provide a good foundation for what to expect from random walk simulations. Let’s now compute (1) to visualize slip functions and to verify the predictions. As a first step, I fix step distance along strike ($\Delta L=100$ m), rigidity ($\mu=3\times10^{10}$ Nm), and down-dip fault width (W=15,000 m). Secondly, I need to find a set of zero mean random variables $\chi_n$ such that the summed variance along the walk (5) equals (15). Many random variables have equal variances and zero mean and all should make quakes that satisfy the scaling relations (22-24). The choice may however, influence the look and feel of the slip functions in subtle ways. The simplest approach assumes that $\chi_n$ are normally distributed with zero mean but with a variance that increases with distance $l=n\Delta L$ along strike as

$$\chi_n = N(0, \sqrt{2qn^{(2q-1)/2}}) \quad (32)$$

For uncorrelated $\chi_n$, the variance along the unrestricted walk (4) is

Figure 7. (Left) Features of restricted random walk quakes with $b=-1$, -3/4, -1/2 and -1/4. The solid lines summarize 1 million random walk quakes. The dashed lines are expected scaling laws (22-24). (Panel A) Number versus Magnitude The slope of these curves is the b-value. (Panel B) Mean slip versus rupture length. (Panel C) Mean slip versus Moment. (Panel D) Mean Length versus Moment. (Right) Features of restricted random walk quakes with $b=-1$. Inclined dashed lines in the bottom panels are the Slip versus Moment relation of Wells and Coppersmith (1994) and the Length versus Moment findings of Kagan (2002). Flat dashed line in Panel C graphs peak height/mean height ratio versus earthquake size.
\[ \sigma^2 (l = n\Delta L) = 2q\Delta u^2 \sum_{i=1}^{n} i^{2q-1} \]
\[ \approx \Delta u^2 \left( \frac{1}{\Delta L} \right)^{2q} \]

as needed by (15). The only free parameters remaining in the simulation are b-value and slip increment \( \Delta u \).

Figure 7 (left) summarizes statistics of one million random walks with \( b = -1/4, -1/2, -3/4 \) and -1. I adjusted slip increment \( \Delta u \) in each case to produce about 1-meter of mean slip on faults of 80 km length. The dashed lines are the predicted scaling laws (22)-(24) and the expected power-law behaviors are all well matched.

The right half of Figure 7 isolates the behaviors for \( b=-1 \) sources. The inclined dashed lines in the bottom two panels graph the independently observed earthquake scaling relations of Wells and Coppersmith (1994)

\[ \log \bar{U}(m) = 0.610 \log M_0(Nm) - 11.75 \]


\[ \log L(km) = 0.315 \log M_0(Nm) - 4.27 \]

Sources with \( b=-1 \) reproduce both of these relations in slope and intercept. The flat dashed line in Panel C (right) plots the peak height to mean height ratio of the simulated ruptures, binned as a function of earthquake moment. As expected, the peak height to mean height ratio depends on b-value but is independent of earthquake size. Computed ratios for \( b = -1/4, -1/2, -3/4 \) and -1 are 2.0, 2.0, 2.3 and 3.2 -- just a bit less than expected from (31).

Figure 8 plots a selection of the random walk ruptures tabulated in Figure 7. The quakes have magnitude 7 to 7.5 drawn from \( b = -1/2, -3/4 \) and -1 sources. You can see that ruptures from lower b-value sources (center and right) are progressively more peaky and skewed than those from \( b=-1/2 \) sources. For comparison, the bottom row of Figure 8 plots the observed slip distributions from Figure 1 scaled somewhat in horizontal and vertical dimensions. Although the

\[ 7.0 < M_w < 7.5 \]

as simulated by a basic random walk (1) with (32) with \( b=-1/2, -3/4 \) and -1.
simulator has yet to be tuned in any way, many of the synthetic slip functions could be mistaken for real.

Alternative Walk. As was said, many possible random variables satisfy (15). Any selection equivalent to basic walk (32) however, produces earthquake slip functions whose potential step to step variations track predictably at each step $n$. To model earthquakes, this feature may be too restrictive. Although the possibilities are still being explored, suppose instead, that the $n$ in (32) is replaced by a random quantity with equal expectation. That is, make the variance of the random variable a random variable itself. From (8a) and (15)

$$E[u(l = n\Delta L)] \approx \sqrt{\frac{\pi}{2}} \sigma(l) \approx \sqrt{\frac{\pi}{2}} \Delta u n^q$$

thus

$$E \left[ \sqrt{\frac{2}{\pi}} \frac{u(n\Delta L)}{\Delta u} \right]^{1/2} \approx n$$

and therefore

$$\chi_n = N(0, \sqrt{2q} \left( \sqrt{\frac{2}{\pi}} \frac{u_{n-1}}{\Delta u} \right)^{(2q-1)/q}) \quad (37)$$

should be a candidate. Potential slip variation in the $n$-th step now scales with the offset at $u_{n-1}$. For $b < -1/2$, step-to-step variations are large where the offset is large and visa versa. For $b = -1/2$ (i.e. $q = 1/2$), (37) reduces to (32). An explicit expression for the variance along the path (like equation 15) is not available for the alternative walk; however, computer simulations reveal that in the mean, the expected rupture shape is

$$E[u_L(l)] = 2U_{\text{peak}} \frac{l^q}{L^{2q}} \left[ L - l \right]^q \quad (38)$$

Alternative walks (37) satisfy the same scaling relations (20, 22-24) as basic walks (32) but with a different $K(q)$

$$K(q) = \sqrt{\frac{\pi}{8}} \left[ \frac{2\Gamma^2(1 + q)}{\Gamma(2 + 2q)} \right]^{1/2} \quad (39)$$

![Figure 9](image)

Figure 9. Earthquake ruptures $7.0 < M_w < 7.5$ as simulated by alternative random walk (37) with $b = -3/4$ and $-1$. Note that the slip functions are more “peaky” (peak slip to mean slip ratios: 2.8 and 4.5 respectively) but that they are unskewed in the mean.
More importantly, because slip (hence slip variance) tapers to zero at both ends of the rupture, offsets (38) are unskewed for any b-value. This may have advantage when fitting real data.

Figure 9 shows a selection random walk earthquakes of magnitude 7 to 7.5 from sources with $b= -3/4$ and -1 using alternative (37). While these walks are slightly peakier than those in Figure 8, they look realistic. In tuning simulations to fit real earthquake observations, blends of the basic and alternative walks might be most suitable.

5. Application: Paleoseismic Rupture Correlation

Even if randomness is a stand in for unknown physics, restricted walk ruptures seem capable of mimicking observed earthquake rates, scaling properties, and "look and feel". Granted this, what purpose can the simulations serve?

A typical situation in paleoseismic work is finding that: 1) Earthquake A broke through Site A $D_1 \pm \Delta_1$ years ago; 2) that Earthquake B broke through Site B $D_2 \pm \Delta_2$ years ago; and that 3) the age limits $D_1 \pm \Delta_1$ and $D_2 \pm \Delta_2$ overlap. The paleoseismic correlation problem amounts to deciding whether rupture events A and B are distinct or are one in the same. The decision often strongly influences earthquake hazard estimates.

In the past, the correlation decision largely hinged on the probability that the age dates were in fact equal. Recently, Biasi and Weldon (2004) proposed a means to improve correlations by blending age overlap probabilities with rupture survival probabilities. If event offset $U$ can be measured at either Site 1 or 2 and the distance between them is known, then a properly tuned random walk simulator can assist in the decision.

The statistical simulator views earthquake correlation as a Gambler's Ruin problem where slip plays the role of money and each km along strike plays the role of a dice toss. The simulator tells us the probability that the rupture will reach from site A to Site B before its slip is played out. For instance, a paleo earthquake rupture is found to have two meters of slip at Wrightwood California. What is the probability that slip runs to zero before the rupture covers the 30 km distance south to Pallet Creek?

Computationally, rupture survival calculations use the simulator parameters ($\Delta u$, $\Delta L$, $W$, $b$ or $q$, and $\chi_n$) tuned to the fault in question. The only difference being that $u_0$ in (1) is set to the slip value $U \gg \Delta u$ observed at one of the paleoseismic sites. After many simulations, the fraction of ruptures that reach distance $L$ are tabulated. Alternatively, if the variance accumulation in the walk $\sigma(L)$ is specified, then equation (6) gives the survival probability analytically. For the basic walk, the rupture survival curves are

$$P_x(U, L) = \text{erf} \left( \frac{U}{\sqrt{2} \sigma(L)} \right)$$

$$= \text{erf} \left( \frac{U}{\sqrt{2} \Delta u(L/L_{\Delta u})^q} \right) = \text{erf} \left( \frac{K(q)}{\sqrt{2} U(L/L_{\Delta u})^q} \right)$$

$$= \text{erf} \left( \frac{K(q)}{\sqrt{2} \Delta u(L/L_{\Delta u})^q} \right)$$

$$= \text{erf} \left( \frac{K(q)}{\sqrt{2} \Delta u} \right)$$
Figure 10 plots rupture survival probabilities for the basic walk with $b=-1$ for $U=0.1, 0.5, 1, 2, \text{ and } 4$ meters. For instance, suppose the distance between the two paleoseismic sites is $L=25$ km, then a rupture with a 10 cm offset at Site A has only 20% chance of reaching Site B. A rupture with a $1/2$ m offset however has a 85% chance of breaking both sites. The last of (40) interprets rupture survival in terms of the mean slip $\bar{U}(L)$ in quakes of length $L$. If the offset $U$ at Site A is large compared to $\bar{U}(L)$, then the rupture will most likely reach the distance $L$ to Site B. No need to be a rocket scientist to understand this conclusion.

Figure 11 plots rupture survival curves for the alternate walk with $b=-1$. For $L=25$ km, a rupture with a 10 (50) cm offset at Site A has only 10% (35%) chance of reaching Site B. The added peakiness in the alternative walk reduces survival probability by about half.

6. Conclusions

Many seismologists are working to construct physically-based earthquake simulators. Physical simulators hold considerable appeal, however they may become so complex that much of the modeling effort is expended in finding a physical basis for essentially random behavior. If certain aspects of earthquake behaviors are random to the extent that real data can constrain them, then a more practical approach may be to embrace the randomness whatever its physical origin.

This article begins to develop and to apply random walk rupture simulations to earthquake issues and to assemble a catalog of observed earthquake slip functions with which to test the simulator. Statistical simulators like these, are intended to complement physical simulators in those applications where they may be better suited. Once calibrated against observed slip functions, statistical simulators
can serve as a scientific tool to aid in the understanding of other characteristics of earthquakes that are not easily, or not yet, measured.

Restricted random walk (1) with standard deviation (6) generates earthquake sets that reproduce observed power law rates of earthquake number versus magnitude as well as diverse earthquake scaling laws such as mean slip versus fault length and moment versus mean slip. By tying together earthquake rates, earthquake scaling laws, and earthquake slip shape and variation through b-value, humble random walks hint at a unified theory of earthquake behavior.

References.


Appendices

A. Error Function

The error function erf(x) smoothly varies from 0 at x=0 to 1 at x=∞ (Figure A1). Also, erf(1)=0.842, erf(2)=0.995, and erf(-x)=-erf(x). For small x, \( \text{erf}(x) \approx 2x / \pi \).

![Figure A1. The error function.](image)

B. Slip variability and rupture survivability

Random walk (1) with \( u_0 = \Delta u \) can be written as a sum or a product of terms

\[
\begin{align*}
    u_n &= \Delta u + \Delta u \sum_{i=1}^{n} \chi_i(0, \sigma_i) \\
    &= \Delta u \prod_{i=1}^{n} \left[ 1 + \left( \frac{\Delta u}{\sigma_i} \right) \chi_i(0, \sigma_i) \right] \\
    &= \Delta u \prod_{i=1}^{n} \left[ 1 + \chi_i \left( \frac{\sigma_i \Delta u}{u_{i-1}} \right) \right] \\
    &= \Delta u \prod_{i=1}^{n} \chi_i \left( 1, \frac{\sigma_i \Delta u}{u_{i-1}} \right)
\end{align*}
\]

The walk terminates at the i-th step when
If $\chi$ is normally distributed, the probability of termination is

$$\chi_i(0, \frac{\sigma_i \Delta u}{u_{i-1}}) < -1$$

For a given offset $u_{n-1}$, the larger the step to step variability $\sigma_i$, the higher the chance of termination. This association gives rise to the inverse relation between slip variability and rupture survivability characterized by b-value.