EFFECTS OF CURVATURE, ASPECT RATIO AND PLAN FORM IN TWO- AND THREE-DIMENSIONAL SPHERICAL MODELS OF THERMAL CONVECTION

GARY T. JARVIS¹,², GARY A. GLATZMAIER³ and VALENTIN I. VANGELOV²

¹ Department of Earth and Atmospheric Science
² Centre for Research in Earth and Space Science York University, North York, Ontario, Canada M3J 1P3
³ Institute of Geophysics and Planetary Physics Los Alamos National Laboratory, Los Alamos, NM 87545, U.S.A.

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Three-dimensional models of thermal convection in a spherical shell are presented for five different cases, each characterized by a unique ratio, $f$, of the radii of the inner and outer bounding surfaces. These solutions are compared to comparable two-dimensional solutions in axisymmetric spherical, cylindrical and Cartesian coordinates. All solutions were obtained with a Rayleigh number of $10^5$, stress free, isothermal boundaries and no internal heating in a constant property Boussinesq fluid of infinite Prandtl number. Similarities and differences between three-dimensional and two-dimensional curvilinear models are discussed in terms of scales and stability of the flow patterns, mean radial temperature profiles and heat transport. It is shown that diagnostic statistics such as mean temperature and Nusselt number may be scaled from one degree of curvature to another for both three- and two-dimensional curvilinear models, provided the aspect ratio and plan form of the flow solutions are comparable. The mean temperature is found to be sensitive to curvature and plan form but not to aspect ratio, while the Nusselt number is found to be sensitive to curvature and aspect ratio but not to the plan form of the flow.

KEY WORDS: Thermal convection, spherical shells.

1. INTRODUCTION

Geophysicists have previously acquired a large body of knowledge and understanding of convection in infinite Prandtl number fluids from extensive laboratory and numerical experiments in plane layers of highly viscous fluids (e.g., Peltier, 1989). With the ever-increasing capacity of modern supercomputers, it is now possible to model, numerically, three-dimensional convection both in plane layers and in the spherical shell geometry appropriate to the Earth's mantle (e.g., Baumgardner, 1985, 1988; Machetel et al., 1986; Cserepes et al., 1988; Glatzmaier, 1988; Houseman, 1988; Bercovici et al., 1989a, b, 1992; Schubert et al., 1990; Travis et al., 1990; Glatzmaier et al., 1990; Gable et al., 1991; Balachandar et al., 1992, 1993; Tackley et al., 1993; Honda et al., 1993; Balachandar and Yuen, 1994; Jordan et al., 1993; Malevsky and Yuen, 1993; Malevsky et al., 1993). Spherical shell models differ from traditional plane layer models in terms of the closed nature and increased aspect ratio of the flow regime.
in addition to the introduction of curvature. It is, therefore, difficult to separate the
effects of curvature, per se, from other associated features. This is true for both two- and
three-dimensional models in curvilinear geometries.

In this paper we compare a suite of three-dimensional models of infinite Prandtl
number thermal convection in a spherical shell with two-dimensional models in plane
layers, cylindrical shells and axisymmetric spherical shells. Our purpose is to determine
how two-dimensional models in various geometries are related to three-dimensional
convection in a spherical shell, and to establish guidelines for applying the simpler
models to the Earth’s mantle. In previous attempts to quantify the relationship between
results obtained in planar and curvilinear geometry Jarvis (1993, 1994) and Vangelov
and Jarvis (1994) compared two-dimensional plane layer model results with two-
dimensional cylindrical and spherical shell models. Here we extend these studies to
comparisons with fully three-dimensional model solutions. Variation in the third
dimension is an additional feature in the latter solutions which tends to obscure the
effects of curvature.

2. MATHEMATICAL FORMULATION

Three-dimensional model results were obtained from the numerical model described
by Glatzmaier (1988). Although this model allows for compressibility in a self gravitating
spherical shell with radial variations of reference state variables, in the present study
these complications were suppressed in an attempt to isolate curvature effects. Thus the
three-dimensional model results shown here pertain to convection in a spherical shell of
an incompressible fluid with constant physical properties, constant gravitational
acceleration and no internal heat sources. Accordingly, the governing pointwise
differential equations describing conservation of mass, momentum and energy, and the
standard linearized equation of state take the form

\[ \nabla \cdot \mathbf{v} = 0, \]  
(1)

\[ 0 = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{v}, \]  
(2)

\[ \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T, \]  
(3)

\[ \rho = \rho_0 [1 - \alpha (T - T_0)], \]  
(4)

(e.g., Landau and Lifshitz, 1959) where \( \mathbf{v} \) is velocity, \( P \) is pressure, \( T \) is temperature
perturbation, \( \mathbf{g} \) is gravitational acceleration, \( t \) is time, and \( \rho, \eta, \kappa \) and \( \alpha \) are the density,
dynamic viscosity, thermal diffusivity and coefficient of thermal expansion, respectively
and \( T_0 \) and \( \rho_0 \) are the reference temperature and density.

The same set of equations is solved in the two-dimensional models but with the
restriction that \( \frac{\partial}{\partial \mathbf{x}_3} = 0 = v_3 \), where in general \( \mathbf{x}_i \) denotes the \( i \)-th spatial coordinate
and \( v_i \) the \( i \)-th component of the velocity field \( \mathbf{v} = (v_1, v_2, v_3) \). Details of the mathematical
formulations for the three-dimensional spherical shell, two-dimensional spherical and
cylindrical shell and two-dimensional plane layer models are given in the Appendix.
parts of this study we examine a restricted range of $\theta$, of angular extent $D = (C - B)$. In the case of the axisymmetric spherical shell model this range may either be centred mid-way between the poles of the coordinate system, in which case the range for $\theta$ is $[B, C] = [(\pi - D)/2, (\pi + D)/2]$, or adjacent to one pole, in which case $[B, C] = [0, D]$. Thus for the spherical shell models, when $D < \pi$ we model an axisymmetric equatorial torus in the former case and a conical plug at the pole in the latter. (In either case when $D = \pi$ the full shell is recovered.) For cases when $\theta \neq 0, \pi/2$ or $\pi$, the stress free boundary condition is physically unrealistic since, in general, shear stresses do not completely vanish between adjacent cells in full-shell multiple-cell solutions. However, this simplification allows us to control aspect ratios for comparisons at different degrees of curvature and/or shell thicknesses.

3. MODEL RESULTS

3.1 Three Dimensional Spherical Convection

In order to examine the effects of curvature on three-dimensional convection models we have generated a suite of five different numerical solutions with varying degrees of curvature as described by the ratio of the radii of the inner and outer boundaries

$$ f = R_1 / R_2. $$

Small values of $f$ correspond to high degrees of curvature. As $f$ increases curvature effects are minimized; in the limit $f \rightarrow 1$ (corresponding to an infinite radius of curvature, or the thin shell approximation) plane layer geometry is recovered. We have considered models with $f = 0.1, 0.3, 0.5, 0.7$ and 0.9, denoted hereafter as models A, B, C, D and E, respectively, all for the same Rayleigh number, $R = 10^5$. Each model was initiated with small random perturbations to an otherwise spherically symmetric conduction profile. Data describing the details of each model are listed in Table I.

Figures 1 and 2 depict great circle cross sections of the temperature perturbation field, $T$, and the lateral distribution of the radial component of velocity, $u_r$, on spherical surfaces at mid-depth in each shell. Model A, with $f = 0.1$, [shown in Figure 1(a)] evolved spontaneously into a steady single-cell flow pattern with rotational symmetry about an axis passing through the centre of the upwelling plume and the centre of the

Table 1 3D Model Characteristics

<table>
<thead>
<tr>
<th>Model</th>
<th>$f$</th>
<th>Radial Resolution</th>
<th>Lateral Resolution</th>
<th>Time Steps</th>
<th>Overturns*</th>
<th>$Nu_{3d}$</th>
<th>$\bar{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.1</td>
<td>65</td>
<td>$32 \times 64$</td>
<td>73,200</td>
<td>9</td>
<td>2.6</td>
<td>0.013</td>
</tr>
<tr>
<td>B</td>
<td>0.3</td>
<td>33</td>
<td>$64 \times 128$</td>
<td>53,000</td>
<td>32</td>
<td>4.4</td>
<td>0.071</td>
</tr>
<tr>
<td>C</td>
<td>0.5</td>
<td>33</td>
<td>$64 \times 128$</td>
<td>30,000</td>
<td>37</td>
<td>6.6</td>
<td>0.182</td>
</tr>
<tr>
<td>D</td>
<td>0.7</td>
<td>33</td>
<td>$64 \times 128$</td>
<td>25,000</td>
<td>30</td>
<td>9.2</td>
<td>0.337</td>
</tr>
<tr>
<td>E</td>
<td>0.9</td>
<td>33</td>
<td>$128 \times 256$</td>
<td>8,000</td>
<td>18</td>
<td>10.4</td>
<td>0.462</td>
</tr>
</tbody>
</table>

* Number of overturns is computed as $[\text{(time elapsed)} \times (2 \times \text{depth})] \times \text{(rms velocity)}.$
Figure 1 Three dimensional spherical convection models. Contours of temperature perturbation, $T$, in great circle cross sections and radial velocity, $v_r$, plotted (in equal area projections) on spherical surfaces at mid-depth of each spherical shell. The temperature perturbation has been scaled so at the outer boundary it is $-f^2$ times its value at the inner boundary. For the temperature plots, solid (broken) contours represent positive (negative) temperature perturbations. For the radial velocity plots, solid (broken) contours represent upward (downward) flow. Rayleigh number is $10^5$ for all cases. Values of $f = R_1/R_2$ are: (a) $f = 0.1$ (model A), (b) $f = 0.3$ (model B) and (C) $f = 0.5$ (model C).

sphere. Model B, with $f = 0.3$ [Figure 1(b)], also evolved into an axisymmetric flow pattern. (Note that without a symmetry breaking mechanism, like rotation, the relationship of the spherical coordinate system to the flow pattern is completely arbitrary.) However, this is not a steady state solution since the flow pattern continues
to slowly evolve. It is, nevertheless, interesting that the initial random perturbations should, at this value of $f$, initially organize themselves into such a near-perfect axisymmetric single cell flow even though this flow pattern does not seem to be stable. We will return to this point in our discussion of (two-dimensional) axisymmetric spherical shell models. Figure 1(c) shows plots of $T$ and $v_r$ at one instant in a fully time-dependent flow for model C, with $f = 0.5$. This solution is characterized by transient plumes emanating from the lower boundary. The parcel of hot fluid near the upper boundary to the right of the north pole is a remnant of a previous hot pulse while that to the left of the north pole remains connected to its source at the inner boundary. Two new instabilities can be seen developing on either side of the current hot plume,
but are being swept with the flow towards the base of the current plume. Thus the
temperature field roughly approximates a two cell flow with polar upwellings and
equatorial downwellings. The distribution of radial velocity, $v_r$, at mid-depth further
illustrates that this two cell pattern is approximately axisymmetric, with narrow
cylinder-like polar upwellings and a sheet-like equatorial downwelling at the equator.
Thus at $R = 10^5$ the three models shown in Figure 1 for $f = 0.1, 0.3$ and 0.5 all tend to
produce axisymmetric flow structures although these are not steady for $f \geq 0.3$.

In Figure 2, sample solutions are shown for models D and E, with $f = 0.7$ and 0.9,
respectively. Although the temperature and velocity fields are only shown at one time
for each of models D and E, it is clear that these are fully three-dimensional
time-dependent flow structures, with separations between upwellings and downwell-
ings comparable to the shell thickness. There is no evident tendency towards rotational
symmetry about any axis. Inspection of Figure 2(a) for the case $f = 0.7$ reveals six major
upwellings and six downwellings in this equatorial great-circle cross-section. The
uniform distribution of upwellings seen in the radial velocity field suggests that any
great circle cross-sectional slice through the temperature field would be representative
of the temperature field. If so, we can infer from Figure 2(a) an angular order of $l = 6
and (in a two-dimensional view) an average angular separation between neighbouring
upwellings and downwellings of $\pi/6$ or 30°. In three dimensions the wavelength, at any
radius, $r$, for a periodic disturbance of order $l$ is

$$\lambda = \frac{2\pi r}{[l(l + 1)]^{1/2}} \approx \frac{2\pi r}{l + \frac{1}{2}}.$$  \hfill (6)

Consequently, the mean angular separation $D$ between upwellings and downwellings is
actually $\pi/(l + \frac{1}{2})$, rather than $\pi/l$. Thus for model D a better estimate is $D = \pi/6.5$ (or
28°).

This visual inference from the spatial fields is corroborated by an examination of the
power spectra of kinetic energy at different times during the model evolution. At all
times the spectra are narrow, peaking in the early stages of the model run at $l = 4, 6,$ or
8. In the final stages, the model spectra consistently peak at $l = 6$.

If we define the aspect ratio, $A$, as the ratio of the lateral arc length between
neighbouring upwellings and downwellings, at mid-depth in the shell, to the shell
thickness, $R_z - R_1$, then

$$A = F D$$  \hfill (7)

where $F$, the dimensionless radial coordinate at mid-depth, is defined as

$$F = \frac{1 + f}{2(1 - f)}.$$  \hfill (8)

For model D, with $f = 0.7$, $F = 2.83$ and since $D = \pi/6.5$, the mean aspect ratio is,
approximately, $A = 1.37$.

A visual analysis of Figure 2(b) for model E, with $f = 0.9$, leads to an estimate of
$l = 30$, and hence $A = 0.98$; essentially unit-aspect-ratio cells. However, the kinetic
energy spectra have broad peaks with significant power in the range $l = 18$ to $l = 25$. 
Because of the ambiguity of the spectral information we will use our visual estimate of \( l = 30 \) for model E while recognizing that is certainly a crude estimate. (Curvature effects are minimal at \( f = 0.9 \) and our conclusions are not significantly influenced by our choice of \( l \) for this model.) In such a thin shell model, the solution appears to be more affected by three-dimensionality than by the curvature or closed nature of the solution domain. The Nusselt number, \( Nu \), (or dimensionless heat flux) computed for this model is \( Nu = 10.4 \). This is very close to the Nusselt number, \( Nu = 10.5 \), obtained for a unit-aspect-ratio in two dimensional plane layer geometry (e.g., Blankenbach et al., 1989). Although the solution domain is closed its lateral extent is so large relative to the shell thickness that the flow field easily adopts its preferred aspect ratio of approximately unity. At the lower values of \( f \) considered here this is not so. The aspect ratios for the five models A to E (beginning with A) are 1.3, 2.0, 1.9, 1.4 and 1.0.

The mean radial temperature profiles for models A – E are shown superimposed in Figure 3. At \( f = 0.9 \), the profile is very similar to the familiar symmetric profile from Bénard convection in plane layers. As \( f \) decreases, the mean dimensionless temperature, \( \bar{T}/\Delta T \), decreases and a pronounced asymmetry develops between the large temperature drop across the inner thermal boundary layer and the small drop across the outer. This asymmetry is required to conduct roughly the same heat through the inner and outer boundaries which have different surface areas. The Nusselt number also decreases from 10.4 at \( f = 0.9 \) to 9.2 at \( f = 0.7 \), 6.6 at \( f = 0.5 \), 4.4 at \( f = 0.3 \) and 2.6 at \( f = 0.1 \).

![Figure 3](image_url)

**Figure 3** Radial profiles of laterally averaged temperature in the three dimensional models A – E shown in Figures 1 and 2. Labels indicate model names and corresponding values of \( f \).
Models A to E represent a set of well resolved vigorously convecting spherical shell models for a range in values of $f$ from 0.1 to 0.9. It is the first such set of three-dimensional numerical solutions designed to isolate the effects of curvature, per se. However, three-dimensional models require large amounts of CPU time on even the largest supercomputers. Therefore, instead of generating a more extensive set of three-dimensional solutions at a variety of Rayleigh numbers, we will attempt to evaluate these models by comparisons with two-dimensional curvilinear models. Specifically, we will compare these results with similar sets of model solutions obtained in axisymmetric spherical and cylindrical shells.

**Model Accuracy and Uncertainties**

The numerical model has been benchmarked against two independent models of H. Harder (pers. comm.) and P. Machete (pers. comm.) for two separate test cases and found to agree to within 0.02%. Further details on the model accuracy are given by Bercovici et al., 1989 (a). Because models B, C, D and E are non-steady, the computed Nusselt varies with time. We find that the Nusselt number in these models (at both inner and outer boundaries) typically varies by less than 0.1 from the values quoted above. For the steady model A, Nusselt numbers computed at the inner and outer boundaries differed by less than 0.001.

### 3.2 Two-Dimensional Axisymmetric Spherical Convection

The axisymmetric spherical convection model of Vangelov and Jarvis (1994) was employed to generate model solutions comparable to the three-dimensional models for $f \geq 0.1$. It was possible to induce steady $I = 1$ flow solutions at both $f = 0.1$ and $f = 0.3$ by applying appropriate initial temperature perturbations and allowing the models to spin up from supercritical conduction states. Contours of temperature and stream function for the axisymmetric model with $f = 0.1$ and $I = 1$ are shown in Figure 4(a). We refer to this two-dimensional (hereafter, 2D) model as A2; comparison with the temperature field shown in Figure 1(a) for the three-dimensional (hereafter, 3D) model A reveals a close similarity. The Nusselt number for the 2D model A2 is 2.8, which agrees to within 8% with the value of 2.6 listed in Table 1 for the steady state 3D model A. This difference can be attributed to the differing resolution of the extreme radial temperature gradients at the inner boundary of the 2D and 3D solutions at $f = 0.1$. Model A2 was obtained on a finite difference grid with uniform radial increments of 1/80, whereas model A was obtained with a radial Chebyshev series truncated at order 65, which has a radial resolution at the inner boundary equivalent to a grid increment of about 1/250. (For models with $f \geq 0.3$, the temperature gradients are less extreme and the 3D models are truncated at order 33 yielding an equivalent radial resolution at the inner boundary of about 1/125. The corresponding 2D models have comparable radial increments of 1/60 at $f = 0.3$ and 1/100 for $f \geq 0.5$. Accordingly, these 2D and 3D models are in closer agreement.)

Temperature and stream function fields are shown in Figure 4(b) for the steady 2D axisymmetric model with $f = 0.3$ and $I = 1$. The temperature field for this model, referred to as B2, is similar to that of the 3D model B shown in Figure 1(b). The Nusselt
number for model B2 is 4.7, about 8% larger than for model B. It was noted above that the 3D model B, although axisymmetric [Figure 1(b)], was a time-dependent solution. It is possible to select a time frame in the evolution of the 2D model B2 for which the Nusselt number matches that of model B. For example Figure 4(c) illustrates an intermediate stage in the evolution of model B2. At the time of Figure 4(c) the two-dimensional model predicts a Nusselt number $Nu = 4.4$ and a mean temperature $\bar{T} = 0.077$ in excellent agreement with the values of 4.4 and 0.071 from the 3D model B.
At $f = 0.5$ we have generated a predominantly $l = 2$ axisymmetric flow solution, model C2, for comparison with the quasi-axisymmetric 3D solution of model C. Both models, C and C2, are time dependent. The temperature and stream function fields shown in Figure 5(a) are for a relatively inactive time following a substantial equatorial downpouring of cold material which has temporarily suppressed the development of boundary layer instabilities in the lower thermal boundary layer. The pattern of isotherms for solution C2, shown in Figure 5(a), resembles that of Figure 1(c). The mean radial temperature profile corresponding to Figure 5(a) is shown in Figure 6(a). The analogous profile from the 3D model solution C is also shown for comparison. Despite the highly time dependent nature of the two models, C and C2, the profiles shown in Figure 6(a) are quite similar. Fluctuations in the model temperature fields produce small variations in the temperature gradients near the boundaries. Consequently, the Nusselt numbers also fluctuate in both time and space. In Figure 6(b) are plotted the Nusselt number, computed as a function of radius, and the separate

![Diagram](image-url)
SPHERICAL MODELS OF THERMAL CONVECTION

\[ f = 0.5 \]

Figure 6  Radial profiles of laterally averaged temperature and Nusselt number at two different times. (a) Mean radial temperature profiles for the 3D spherical model C\((f = 0.5)\), labelled 3D, and the 2D axisymmetric spherical model C2, labelled Ax. (b) Nusselt number evaluated at each radial coordinate, \( Nu \), and the separate conductive and advective contributions to \( Nu \), labelled "cond" and "adv" respectively. At each radial coordinate \( Nu = Cond + Adv. \) (c) and (d): Same as (a) and (b) but at a later time in the evolution of model C2.

advective and conductive contributions to it. In a steady state the Nusselt number would not vary with \( r \). For model C2 at this instant the Nusselt number has values of \( Nu = 6.47 \) at the inner boundary, and \( Nu = 7.02 \) at the outer boundary. We take the average of these two values, \( Nu = 6.7 \), to be representative of model C2. This value agrees to within 2\% of the value of \( Nu = 6.6 \) listed in Table 1 for the 3D model C. This close agreement is in part due to the similarity of the model C2 temperature field which was selected to compare with that for model C. In contrast, Figures 6(c) and 6(d) show mean temperature and Nusselt number profiles for model C2 at a later time during a surge of advective heat transfer. At this time the Nusselt number has values of 6.9 at the inner surface, and 7.9 at the outer surface. These values do not compare favourably with model C, and indicate that large heat flow fluctuations can occur in two cell
time-dependent flows. (At higher Rayleigh numbers, more cells occur (Glatzmaier, 1988) and self cancellation of individual cell fluctuations tends to reduce the total variation.)

For values of $f = 0.7$ and $f = 0.9$ there is no axisymmetric character to the three-dimensional models. Nevertheless, we compare the two-dimensional axisymmetric models to the fully three-dimensional models attempting to match aspect ratios in the models being compared. In the discussion of model D it was established that the mean aspect ratio of the cells at $R = 10^5$ and $f = 0.7$ was 1.37. We have run two axisymmetric models with $A = 1.37$ for comparison. The first model, D2, consists of an axisymmetric torus centred on the equator of the coordinate system such that the range of colatitude is $\theta = [(\pi - D)/2, (\pi + D)/2]$. Vangelov and Jarvis (1994) have argued that such a torus is most representative of two-dimensional rolls in spherical geometry. However, inspection of Figure 2(a) indicates that the three-dimensional structure of the individual cells is similar to that produced by the two-dimensional axisymmetric model in the near vicinity of the poles. Therefore, a second axisymmetric model, D3, was run centred on a polar upwelling with a range in colatitude of $\theta = [0, \pi/6.5]$. The Nusselt numbers predicted by these models are $Nu = 9.4$ for models D2 and $Nu = 9.2$ for model D3. These compare favourably with the three-dimensional model D for which $Nu = 9.2 \pm 0.1$. Model D3 compares most favourably with a difference in predicted $Nu$ of less than 1%, and a similar flow structure consisting of a narrow cylindrical upwelling surrounded by a sheet-like ring of downwelling. Temperature and stream function contours for this model are shown in Figure 7.

At $f = 0.9$ the effects of curvature are almost negligible in the two-dimensional models (Jarvis, 1993, 1994; Vangelov and Jarvis, 1994). Inspection of Figure 2(b) for

![Figure 7](image-url)  

**Figure 7** Temperature and stream function fields for 2D axisymmetric shell model D3 with $R = 10^5$, $f = 0.7$ and $A = 1.37$. 

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model E suggests that curvature effects are also small in three-dimensional models at \( F = 0.9 \). Model E predicts \( A \approx 1 \) and \( Nu = 10.4 \). This value of the Nusselt number deviates by less than 1% from the known value of 10.53 for two-dimensional unit-aspect-ratio plane layer convection rolls. Vangelov and Jarvis (1994) found \( Nu = 10.5 \) for a two-dimensional equatorial torus of unit aspect ratio and \( f = 0.9 \) in their axisymmetric spherical model. For a unit aspect ratio flow centred on the pole with polar upwelling (model E2), we find \( Nu = 10.42 \) in good agreement with the three-dimensional result of model E.

### 3.3 Two-Dimensional Cylindrical Models

Two-dimensional models in cylindrical polar coordinates \((r, \theta, z)\), with the z-axis horizontal and \( \partial / \partial z = 0 \), produce flow fields in the \( r - \theta \) plane which are similar in some respects to the axisymmetric spherical models. Curvature effects are included, the solution domain is of finite extent and upwellings and downwellings occur as sheets similar to those in the axisymmetric sphere at positions distant from the poles. The cylindrical models differ from the axisymmetric spherical models in that for a given \( f \) the effects of curvature are less pronounced in cylindrical geometry. However, the effects of curvature are similar for cylindrical and spherical shells which have the same values of \( f_a \), the ratio of the surface areas of the bounding surfaces (Vangelov and Jarvis, 1994). Sample temperature and stream function fields for convection in a cylindrical shell with \( f = 0.5 \) and \( l = 2 \), model C3, are shown in Figure 5(b). In two-dimensional cross section this field looks similar to either the 2D axisymmetric field for model C2 shown in Figure 5(a) or the 3D field for model C shown in Figure 1(c). However, unlike models C and C2 the full three-dimensional flow represented by the 2D cylindrical model C3 has neither concentrated upwellings nor curvature in the direction orthogonal to the plane of the figure.

Jarvis (1993, 1994) has shown that the Nusselt number, \( Nu \), and mean temperature, \( \bar{T} \), in cylindrical shells may be accurately predicted for steady flow fields at any \( f \geq 0.3 \) in terms of the corresponding values in a plane layer provided the aspect ratio of the convection cells (lateral arc length at mid depth: shell thickness) is the same. The predicted values \( Nu^p(f) \) and \( \bar{T}^p(f) \) are given by the expressions

\[
Nu^p(f) = -\frac{2^{4/3} \ln f}{(1-f)(1+f^{-3/4})} Nu(1) \tag{9}
\]

and

\[
\bar{T}^p(f) = \frac{2}{1+f^{-3/4}} \bar{T}(1), \tag{10}
\]

where \( Nu(1) \) is the Nusselt number and \( \bar{T}(1) \) the mean temperature in a plane layer (i.e., \( f = 1 \)) flow field of the same aspect ratio. The dependence of \( Nu^p(f) \) and \( \bar{T}^p(f) \) on aspect ratio is contained implicitly in \( Nu(1) \) and \( \bar{T}(1) \). For convenience we may re-write (9) and
where \( f_a \) is the ratio of the bounding surface areas, and \( G_c \) and \( H \), representing the fractions in (9) and (10) respectively, are simple geometric scaling factors which depend only on \( f \). (The subscript \( c \) refers to cylindrical shell geometry.) In (11) and (12) we have written the argument of \( G \) and \( H \) as \( f_a \), the ratio of the surface areas of the inner and outer bounding surfaces, rather than \( f \). For cylindrical geometry this is of no consequence since \( f_a = f \). However, in spherical geometry \( f_a = f^2 \) and Vangelov and Jarvis (1994) have found that cylindrical shell and spherical shell models give similar results for the same values of \( f_a \) rather than \( f \). Thus (11) and (12) may also be used to predict the Nusselt number and mean temperature in spherical shell geometry if \( f_a \) is used in place of \( f \) on the right-hand side only of (9) and (10).

One deficiency in using \( G_c(f_a) \) for spherical shells is that the Nusselt number is defined as the total heat flow normalized by the theoretical conductive heat flux under the same conditions. The dimensionless theoretical conductive flux for cylindrical geometry, \( F_{\text{cond}} = -1/[r \ln(f)] \), has been incorporated into (9) and hence into \( G_c(f_a) \). Replacing this term with \( F_{\text{cond}} = f/[r^2(1-f)^2] \) for spherical geometry (evaluated at \( r = R_2 \)) results in the following expression which is more appropriate for a spherical shell,

\[
Nu^p(f) = \frac{2^{4/3}}{f_a^{1/2}(1 + f^{-3/4})^{1/3}} Nu(1) = G_s(f_a) Nu(1),
\]

where \( G_s(f_a) \) represents the term in brace brackets, and the subscript \( s \) refers to spherical geometry.

For purposes of comparison with the spherical models, a series of 5 unit-aspect-ratio cylindrical shell models were run with \( f = f_a \) equal to 0.01, 0.09, 0.25, 0.49 and 0.81, corresponding to the respective values of \( f_a \) of the 3D spherical models. The resulting Nusselt numbers, used in constructing Table 3, are discussed in the following section.

4. DISCUSSION OF RESULTS

Our initial discussion will centre on the data listed in Table 2, for each of models A through E. The first seven columns list, respectively, the model name; its value of \( f = R_1/R_2 \); \( F \), the dimensionless radius at mid-depth; \( l \), the dominant degree of the convective pattern; \( D \), the mean angular separation between upwelling and downwelling flow; \( A \), the aspect ratio; and \( Nu_{3D} \), the Nusselt number evaluated at the outer surface of the 3D spherical shell model solutions.
Table 2  Geometric Factors and Nusselt Numbers

<table>
<thead>
<tr>
<th>Model</th>
<th>$f$</th>
<th>$F$</th>
<th>$l$</th>
<th>$D = \pi(l + 0.5)$</th>
<th>$A = FD$</th>
<th>$Nu_{3D}$</th>
<th>$G_s(f)$</th>
<th>$Nu^p(f)$</th>
<th>$s(A)$</th>
<th>$s(A) \cdot Nu^p(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.1</td>
<td>0.611</td>
<td>1</td>
<td>2.094</td>
<td>1.28</td>
<td>2.6</td>
<td>0.242</td>
<td>2.55</td>
<td>0.961</td>
<td>2.5</td>
</tr>
<tr>
<td>B</td>
<td>0.3</td>
<td>0.929</td>
<td>1</td>
<td>2.094</td>
<td>1.95</td>
<td>4.4</td>
<td>0.617</td>
<td>6.50</td>
<td>0.776</td>
<td>5.0</td>
</tr>
<tr>
<td>C</td>
<td>0.5</td>
<td>1.50</td>
<td>2</td>
<td>1.257</td>
<td>1.88</td>
<td>6.6</td>
<td>0.841</td>
<td>8.86</td>
<td>0.791</td>
<td>7.0</td>
</tr>
<tr>
<td>D</td>
<td>0.7</td>
<td>2.83</td>
<td>$\approx$6</td>
<td>0.483</td>
<td>1.37</td>
<td>9.2</td>
<td>0.954</td>
<td>10.0</td>
<td>0.940</td>
<td>9.4</td>
</tr>
<tr>
<td>E</td>
<td>0.9</td>
<td>9.50</td>
<td>$\approx$30</td>
<td>0.103</td>
<td>$\approx$0.98</td>
<td>10.4</td>
<td>0.996</td>
<td>10.5</td>
<td>1.00</td>
<td>10.5</td>
</tr>
</tbody>
</table>
4.1 The Role of Aspect Ratio

Vangelov and Jarvis (1994) showed that the Nusselt numbers in 2D axisymmetric, unit-aspect-ratio, equatorial tori could be predicted with reasonable accuracy with the expression given in (13). At \( R = 10^5 \), \( \text{Nu}(1) \) is known to be 10.533 in 2D unit aspect ratio flows (e.g., Blankenbach et al., 1989). Values of the scaling factor \( G_\epsilon \) required in (13) are listed in the eighth column of Table 2. Multiplying each of these values by 10.53 yields the predicted Nusselt numbers \( \text{Nu}^p(f) \) appearing in the ninth column of Table 2. These predicted values do not agree well with the 3D model results \( \text{Nu}_{3D} \) appearing in column 7. However, it must be remarked that the aspect ratios of the 3D models (column 6) are all different. For cylindrical shells Jarvis (1993, 1994) has shown that the dependence of \( \text{Nu}(f) \) on aspect ratio is very similar to that of \( \text{Nu}(1) \) for \( f \geq 0.3 \). This is assumed to be true for spherical shells in the derivation of (13). Consequently, each of the data listed in column 8 need to be scaled by a factor \( s(A) \) which accounts for the fact that \( A \neq 1 \). These factors were obtained by running a series of plane layer models with \( R = 10^5 \) and various values of \( A \) between 1.0 and 1.8. Beyond \( A = 1.8 \), steady model solutions could not be obtained and values of \( \text{Nu} \) at \( A = 1.88 \) and 1.95 were estimated with a quadratic equation fit to the Nusselt numbers at \( A = 1.6 \), 1.7 and 1.8. The resulting scaling factors \( s(A) \), given by the ratios \( \text{Nu}(A)/\text{Nu}(A = 1) \), appear in column 10 of Table 2. Column 11 shows the result of multiplying our first simple prediction, \( \text{Nu}^p \) (column 9), by \( s(A) \). This final column of data in Table 2 compares much more favourably with \( \text{Nu}_{3D} \) listed in column 7. We emphasize that the data listed in column 11 are simply the product of a constant, 10.53 (derived for unit-aspect-ratio plane layer convection), a geometrical scaling factor \( G_\epsilon(f) \) which depends only on \( f \) (the ratio of the radii of the bounding surfaces), and a second scaling factor \( s(A) \), which describes the variation of Nusselt number with aspect ratio in plane layer convection. Given the simplicity of this prediction as compared to the complex time-dependent nature of the 3D models, it is surprising that the agreement is so close. The largest discrepancy occurs not for the smallest \( f \) but for the largest value of aspect ratio, \( A = 1.95 \), found in model B. The prediction process is demonstrated graphically in Figure 8 which shows plots of \( \text{Nu} \) versus \( f \). Beginning at the right hand margin \( (f = 1) \) the plane layer Nusselt numbers are selected for aspect ratios corresponding to the 3D models. These values are then projected to lower \( f \) along the solid curves which are determined from (13). It is clear that no single curve could fit all five 3D model predictions because of the dependence on aspect ratio. The best fitting projections to the 3D results occur for \( f = 0.9 \) and \( f = 0.1 \), both of which have aspect ratios close to unity; the worst fitting projections are for \( f = 0.3 \) and 0.5, the cases with the largest aspect ratios (of approximately two).

The predicted Nusselt numbers listed in column 11 of Table 2 are also quite similar to the values obtained for the 2D axisymmetric spherical shell models which were compared to the 3D models in Section 3.2, and presented in Figures 4 to 7 (namely, A2, B2, C2, D3 and E2). Nusselt numbers for these axisymmetric models are juxtaposed with those of the 3D models and the predicted values (column 11 of Table 2) in the first four columns of Table 3. In Table 3 the 3D models are labelled \( \text{Nu}_{3D}(f, A) \) to indicate, explicitly, their dependence on both \( f \) and \( A \). Similarly, the predictions from column 11 of Table 2 are here denoted \( \text{Nu}^p(f, A) \) and the 2D axisymmetric results are labelled \( \text{Nu}_{\text{Ax}}(f, A) \). The close agreement between values of \( \text{Nu}_{\text{Ax}}(f, A) \) and \( \text{Nu}_{3D}(f, A) \) is
attributed to our having matched both the degrees of curvature and aspect ratios in the 2D and 3D models.

4.2 The role of plan form

An unexpected result is that the Nusselt number is not sensitive to the basic geometry, or plan form, of the flow fields. The values of $\text{Nu}_{\text{Axi}}(f, A)$ were all obtained with 2D flow
models which featured a concentrated cylindrical upwelling at the pole \( \theta = 0 \) and broad sheet-like downwellings at \( \theta \leq \pi/2 \). This plan form was selected in order to best match the patterns of isotherms in the 3D models. However, 2D axisymmetric flow structures centred on the equator in the form of equatorial tori also yield Nusselt numbers in agreement with the 3D models, provided both \( f \) and \( A \) are accounted for. This is illustrated in columns 5 and 6 of Table 3. In column 5 are listed the model results from the Vangelov and Jarvis (1994) study of unit-aspect-ratio equatorial tori in 2D axisymmetric spherical geometry. We denote these values \( \mathcal{N}_u^{2D}(f, A = 1) \) to emphasize explicitly that they are a function of \( f \) only. To account for the different aspect ratios of the corresponding 3D models these values are scaled by the factor \( s(A) \) (from column 10 of Table 2) to yield the data in column 6 of Table 3 (labelled \( s(A) \cdot \mathcal{N}_u^{2D} \)). These data compare equally well with \( \mathcal{N}_u^{3D}(f, A) \) as do the \( \mathcal{N}_u^{3D}(f, A) \) data of column 4 (from models with concentrated polar upwellings). A further indication of the insensitivity of \( \mathcal{N}_u \) to flow geometry comes from the five 2D cylindrical flow models referred to above, run for the same values of \( f \) as the 3D models. These models were run with unit aspect ratios and consequently their Nusselt numbers, denoted \( \mathcal{N}_u^{3D}(f, \cdot) \), do not agree with those of the 3D models, \( \mathcal{N}_u^{3D} \). However, when scaled by \( s(A) \) to account for the different aspect ratios of the 3D models the resulting values agree remarkably well with \( \mathcal{N}_u^{3D} \). The data from the scaled cylindrical values are listed in the final column of Table 3 under the heading \( s(A) \cdot \mathcal{N}_u^{3D}(f, \cdot) \). Thus the flow geometry (i.e., upwelling cylindrical plumes versus upwelling sheets) does not appear to have a significant effect on the Nusselt number.

This conclusion is consistent with the observation that in high Rayleigh number plane layer models the Nusselt number is the same for either 2D or 3D simulations despite the difference in the thermal structure of 2D sheet-like, and 3D cylindrical, upwellings and downwellings (e.g., Malevsky and Yuen, 1993). Further evidence of the insensitivity of \( \mathcal{N}_u \) to the plan form of upwellings and downwellings comes from 2D models of axisymmetric cylindrical upwellings in a plane layer surrounded by a sheet-like downwelling. Kiefer and Hager (1992) describe such a model formulated in cylindrical coordinates with z-axis vertical. The flow fields are obtained in the \( r-z \) plane with the assumption that \( \partial / \partial \theta = 0 \). We will refer to this as the “pill box” model geometry. For a unit aspect ratio and \( R = 10^5 \), they obtain a Nusselt number of 10.42 which differs from the 2D Cartesian result (with symmetric upwelling and downwelling sheets) by less than 1%. Thus, in general, model Nusselt numbers appear to be sensitive to aspect ratio but not to plan form.

In contrast, the central temperatures of convecting layers are not sensitive to aspect ratio (e.g., Jarvis, 1994) but are sensitive to the plan form of the flow. For conventional 2D plane layer Bénard convection rolls, with sheet-like upwellings and downwellings, the mean dimensionless temperature is \( \bar{T} = 0.5 \). However, for axisymmetric cylindrical upwellings in the plane layer pill box geometry the dimensionless mean temperature is \( \bar{T} \approx 0.35 \) (Kiefer and Hager, 1992; Keifer, pers. comm.). Thus the term \( \bar{T}(1) \) in (12) will vary with planform between 0.35 for cylindrical upwellings and 0.5 for sheet-like upwellings.

All of the 2D axisymmetric spherical shell models centred on the pole at \( \theta = 0 \) have a focused cylindrical upwelling at the pole. Those with \( f \geq 0.5 \) have polar upwellings surrounded by diffuse downwelling sheets; these are spherical analogues of the plane
layer pill box geometry and, therefore, could be expected to have mean temperatures given by $\bar{T}_p = 0.35 H(f)$. This is verified graphically in Figure 9 where radial profiles of laterally averaged temperature are plotted separately for different values of $f$ between $f = 0.3$ and 0.9. Superimposed on each graph is the corresponding mean radial temperature profile from the 3D spherical model. Two vertical bars on each graph indicate values of $\bar{T}_p = 0.35 H(f)$ and $\bar{T}_p = 0.5 H(f)$, the first being appropriate to flow fields dominated by cylindrical upwellings and broad sheet-like downwellings, and the second appropriate to flow fields with symmetric sheet-like upwellings and downwellings. Profiles from the 2D axisymmetric models (labelled Ax in Figure 9) have central temperatures very close to $\bar{T}_p = 0.35 H(f)$ for all $f$. For $f = 0.3$ and 0.5 it is clear that the 3D model profiles (labelled 3D on Figure 9) and the axisymmetric model profiles closely agree with each other. (The same is true for $f = 0.1$ - not shown.)

**Figure 9** Comparisons of mean radial temperature profiles from 3D spherical models, labelled 3D, and 2D axisymmetric spherical models, labelled Ax, for different values of $f$: (a) $f = 0.3$, (b) $f = 0.5$, (c) $f = 0.7$, (d) $f = 0.9$. The two vertical bars in each frame represent the scaled mean temperatures for convection in plane layers. The bar at the lower temperature in each frame is scaled from the pill-box value of 0.35 while the bar at the higher temperature is scaled from the Cartesian-roll value of 0.50. In both cases scaling is according to (12).
Furthermore, for $f = 0.5$ the mean temperature, 0.176, agrees to within a few percent with the scaled pill box mean temperature, 0.183. Figure 1 shows that the 3D model C with $f = 0.5$ is dominated by focused cylindrical upwellings and broad downwelling sheets.

The 3D model temperature profiles for $f = 0.7$ and 0.9 in Figure 9 depart from the axisymmetric model profiles and their central temperatures move towards the scaled value of the Cartesian-geometry mean temperature. Inspection of Figure 2 reveals a decreasing dominance of the upwelling plumes and more focusing of downwelling sheets as $f$ increases above 0.5. At $f = 0.7$ several regions of upwelling are seen to be connected. At $f = 0.9$ the scale of both upwellings and downwellings is so small that it is difficult to distinguish regions of positive radial velocity from regions of negative. Thus, although the plan form is not one of symmetric upwelling and downwelling sheets, the dominance of cylindrical upwellings, apparent at low $f$, is lost in the thin shell model at $f = 0.9$. Accordingly, the mean temperature of the convecting shell is best predicted by scaling the Cartesian-geometry value of 0.5 to $f = 0.9$ as $T_f = 0.5 H(0.81)$. Taking plan form into account we would predict, from (12), mean temperatures of 0.021, 0.099, 0.183, 0.369 and 0.461, for $f = 0.1, 0.3, 0.5, 0.7$ and 0.9, respectively. For the two cases $f = 0.5$ and 0.9, these values agree closely with the values of $T$ listed for the 3D models in Table 1. At $f = 0.7$ plan form is intermediate and $T$ lies between the scaled pill box mean temperature of 0.259 and the scaled 2D Cartesian rolls mean temperature of 0.369. At $f \leq 0.3$ both upwellings and downwellings occur as cylindrical plumes (at opposite poles). This geometry does not have a plane layer analogue and, consequently, neither of the scaled pill box or Cartesian roll mean temperatures quoted above agree with the 3D model temperatures.

5. CONCLUSIONS

We have generated numerical models of convection with various degrees of curvature in 3D spherical shells, 2D axisymmetric spherical shells and 2D cylindrical shells. We have compared the Nusselt numbers, radial temperature profiles and mean temperatures of the convecting shells produced in the various geometries with each other and with the corresponding parameters known for plane layer geometry. Different results in the various geometries can be accounted for in terms of the different plan forms of the flow structure, mean aspect ratios of the cells and ratios of the surface areas of the bounding surfaces. Both Nusselt number and mean temperature depend on the degree of curvature. However, the Nusselt number is found to be relatively insensitive to plan form but sensitive to the aspect ratio, while the mean temperature is relatively insensitive to aspect ratio but sensitive to plan form. Provided curvature, aspect ratio and plan form are accounted for it is possible to reconcile the 3D spherical shell model solutions with those formulated in two dimensions. Moreover, for $f > 0.3$ the Nusselt numbers and mean temperatures of the 3D models can be predicted in terms of the corresponding values found in 2D plane layer geometry and simple geometrical scaling factors which depend only on $f$, the ratio of the radii of the two bounding surfaces. These conclusions should facilitate the application of results obtained in two-dimensional models to the three-dimensional Earth.
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References


APPENDIX: MATHEMATICAL FORMULATIONS

1. Three-dimensional models

For the case of an incompressible fluid with constant physical properties, constant gravitational acceleration and no internal heat sources, the velocity field may be expressed in terms of a stream function \( \Psi \) as

\[
v = \nabla \times \Psi \tag{14a}
\]

such that

\[
\Psi = \nabla \times (W \mathbf{r}) \tag{14b}
\]

where \( W \) is a scalar field referred to as the poloidal mass flux and \( \mathbf{r} \) is a unit radial vector. Thus velocity components are derived from the numerical solution of the scalar field \( W \). The temperature and poloidal mass flux fields are each expanded laterally in a surface spherical harmonics series truncated at angular degree and order \( l_{\text{max}} \) and radially in a Chebyshev series truncated at radial order \( n_{\text{max}} \). Constant temperature, stress free boundary conditions are applied at the inner and outer boundaries, \( R_1 \) and \( R_2 \), respectively. Given an initial random temperature field, a semi-implicit time-integration scheme is used, with a spectral transform method and Chebyshev collocation, to march the temperature field forward in time.
2. Two-dimensional models

In all of the two-dimensional models, equations (1) to (4) are non-dimensionalized using as characteristic values for linear distance, temperature, velocity and density: \( d = R_2 - R_1 \), the shell thickness; \( \Delta T \), the imposed temperature difference across the depth of the convecting shell; \( V = \rho_0 g \alpha \Delta T d^2/\eta \), the advective velocity scale (e.g., McKenzie et al., 1974); and \( \rho_0 \), the reference density \( \rho(T_0) \). The two-dimensional models all generate numerical solutions to the same governing equations as the three-dimensional models [equations (1) to (4)] but with the imposed conditions \( \partial/\partial x_3 = 0 = v_3 \), where in general \( x_i \) denotes the \( i \)-th spatial coordinate and \( v_i \) the \( i \)-th component of the velocity field \( \mathbf{v} = (v_1, v_2, v_3) \).

For incompressible two-dimensional solutions, in which \( \partial/\partial x_3 = 0 \), the velocity field may be expressed in terms of single component stream function, \( \Psi = (0, 0, \psi^*) \), and vorticity \( \omega = (0, 0, \omega) \), fields directed orthogonal to the plane containing the fluid motion. Thus from (14a) we obtain velocity components as

\[
v_1 = \left(1/h_2 h_3\right) \partial (h_3 \psi^*) / \partial x_2
\]

and

\[
v_2 = - \left(1/h_1 h_3\right) \partial (h_3 \psi^*) / \partial x_1,
\]

where \( h_i \) represents the \( i \)-th metric coefficient of the coordinate system (e.g., Batchelor, 1967). The stream function and vorticity fields are related as

\[
\omega = \nabla \times (\nabla \times \Psi) = - \nabla^2 \Psi.
\]

The velocity field as defined by (14) and (15) automatically satisfies the continuity equation (1). In two-dimensions only, (1) may also be satisfied by scaling the single component stream function such that

\[
\Psi = (0, 0, \psi^*) = (0, 0, \psi / h_3).
\]

In terms of \( \psi \), (15) simplify to

\[
v_1 = (1/h_2 h_3) \partial \psi / \partial x_2,
\]

and

\[
v_2 = - (1/h_1 h_3) \partial \psi / \partial x_1.
\]

Equations (18) are commonly used in two-dimensional axisymmetric spherical geometry (e.g., Zebib et al., 1980; Solheim and Peltier, 1990), as well as in cylindrical and Cartesian geometry (Jarvis, 1993).
Equations (1) to (4) may be expressed in terms of vorticity, stream function and temperature as

\[ \nabla^2 \omega = \frac{1}{h^2} \frac{\partial T}{\partial x_2} e_3, \tag{19} \]

\[ \nabla^2 \Psi = -\omega, \tag{20} \]

\[ \frac{\partial T}{\partial t} = -(1/h_2)J_{12}(T,\Psi) + R^{-1} \nabla^2 T, \tag{21} \]

where \( J_{12} = (V_1 T \nabla_2 \Psi - \nabla_2 T \nabla_1 \Psi) \) is the Jacobian with respect to the first two components of the gradient operator \( \nabla = (\nabla_1, \nabla_2, \nabla_3) \), \( e_3 \) is a unit vector in the \( x_3 \)-direction, \( R = (g \alpha \Delta T d^3)/(\kappa v) \) is the Rayleigh number, a dimensionless constant, and \( v = \eta/\rho_0 \) is the kinematic viscosity. Note, the same definition of the Rayleigh number is used for the three-dimensional spherical shell model.

Equations (19) to (21) are valid for any orthogonal coordinate system, provided \( \partial/\partial x_3 = 0 \); expressed in vector format they take the same form as the familiar Cartesian scalar equations (e.g. McKenzie et al., 1974). In a plane layer Cartesian coordinate system \((z, x, y)\), the metric coefficients are \( h_1 = h_2 = h_3 = 1 \), and thus (18) yield

\[ v_x = -\frac{\partial \Psi}{\partial z}, \quad v_z = \frac{\partial \Psi}{\partial x}; \tag{22a} \]

in cylindrical coordinates \((r, \theta, z)\), \( h_1 = h_3 = 1 \) and \( h_2 = r \), and (18) yield

\[ v_r = (1/r)\frac{\partial \Psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \Psi}{\partial r}; \tag{22b} \]

and in spherical coordinates \((r, \theta, \phi)\), \( h_1 = 1 \), \( h_2 = r \) and \( h_3 = r \sin \theta \), and hence

\[ v_r = \left[1/(r^2 \sin \theta)\right] \frac{\partial \Psi}{\partial \theta}, \quad v_\theta = -(1/r \sin \theta) \frac{\partial \Psi}{\partial r}. \tag{22c} \]

Equations (19) to (21) each yield one scalar equation. In Cartesian coordinates these take the familiar form

\[ \nabla^2 \omega = \partial T / \partial x, \tag{23a} \]

\[ \nabla^2 \Psi = -\omega, \tag{24a} \]

\[ \frac{\partial T}{\partial t} = -J_{x,z}(T,\Psi) + R^{-1} \nabla^2 T. \tag{25a} \]

In cylindrical coordinates these are

\[ \frac{\partial^2 \omega}{\partial r^2} + (1/r)\frac{\partial \omega}{\partial r} + (1/r^2)\frac{\partial^2 \omega}{\partial \theta^2} = (1/r)\frac{\partial T}{\partial \theta}, \tag{23b} \]

\[ \frac{\partial^2 \Psi}{\partial r^2} + (1/r)\frac{\partial \Psi}{\partial r} + (1/r^2)\frac{\partial^2 \Psi}{\partial \theta^2} = -\omega, \tag{24b} \]

\[ \frac{\partial T}{\partial t} = -J_{r,\theta}(T,\Psi) + R^{-1} \nabla^2 T. \tag{25b} \]
In spherical coordinates (19) to (21) yield

\[ \nabla^2 \omega - \omega / (r^2 \sin^2 \theta) = \left( \frac{1}{r} \right) \partial T / \partial \theta, \tag{23c} \]

\[ \frac{\partial^2 \psi}{\partial r^2} + \left( \frac{1}{r^2} \right) \frac{\partial^2 \psi}{\partial \theta^2} - \left( \cot \theta / r^2 \right) \frac{\partial \psi}{\partial \theta} = - \omega r \sin \theta, \tag{24c} \]

\[ \frac{\partial T}{\partial t} = - \left( \frac{1}{r^2 \sin \theta} \right) J_{\phi}(T, \psi) + R^{-1} \nabla^2 T. \tag{25c} \]

With appropriate boundary conditions on \( T, \omega \) and \( \psi \), and an initial \( T \) field, each set of (23) to (25) may be solved cyclically, marching the temperature field forward in time with the appropriate equation (25) and obtaining at each time step the corresponding vorticity and stream function fields, \( \omega \) and \( \psi \), from the corresponding equations (23) and (24).

Constant temperatures \( T = \Delta T \) and \( T = 0 \) are imposed at \( r = R_1 \) and \( R_2 \) respectively. In dimensionless form \( R_2 - R_1 = 1, \ R_2 = (1 - f)^{-1} \) and \( R_1 = f(1 - f)^{-1} \). Vanishing normal velocities and shear stresses at the inner and outer boundaries imply \( \psi = 0 \) and either \( \omega = 0 \), for Cartesian geometry, \( \omega = - (2/r) \partial \psi / \partial r \), for cylindrical geometry, or \( \omega = - (2/r^2 \sin \theta) \partial \psi / \partial r \), for spherical geometry, at both \( r = R_1 \) and \( r = R_2 \). Because of the coupling of \( \omega \) and \( \psi \) in the boundary conditions for the cylindrical and spherical shell models, an initial trial solution for \( \omega \) is obtained using values of \( \psi \) at the boundaries from the previous time step. Revised solutions for the complete \( \omega \)- and \( \psi \)-fields are then obtained by solving (23) and (24) iteratively (at each time step) until convergence is obtained such that the boundary condition is achieved. The curvilinear models span the angular range \([B, C]\) with boundary conditions at \( \theta = B \) and \( \theta = C \) of vanishing normal velocities, normal heat flux and tangential shear stresses, or \( \omega = \psi = \partial T / \partial \theta = 0 \). These conditions provide a self-contained flow regime with no net lateral transfer of heat or momentum.

The finite difference approximation to (23)–(25) follows the approach described by Jarvis and Peltier (1982) for plane layers, by Jarvis (1993) for cylindrical shells and by Solheim and Peltier (1990) for spherical shells.