Astronomy 112: The Physics of Stars

Class 8 Notes: Nuclear Chemistry in Stars

In the last class we discussed the physical process of nuclear fusion, and saw how rate coefficients for nuclear reactions are calculated. With this understanding in place, it is possible to examine to study which reactions chains are actually important in generating energy and driving the evolution of stars. Those reaction chains and their properties will be the topic of this class.

I. Characterizing Reactions

To begin, it is useful to review what we know about nuclear reactions and what we want to know. A first principle of importance is that reaction rates are very, very sensitive to temperature, so that they can go very rapidly from negligible to huge. As a result, there is usually only a narrow range of temperatures where a reaction can occur for a period of time comparable to t_{KH}, the thermal timescale of the star. At lower temperatures the reaction would produce negligible energy, and at higher temperatures it would rapidly consume all the available fuel in a time much less than t_{KH}. Because the temperature windows where reactions occur at moderate rates are narrow, it is usually (though not always) the case that, in a given region within a star, there is only one reaction (or reaction chain) that occurs over an extended period in a given part of a star.

By figuring out at what temperature such moderate reaction rates set in, we can assign a characteristic ignition temperature T_i to a given reaction or reaction chain. This ignition temperature is one of the basic things we want to know about a nuclear reaction. We would also like to know the reaction rate coefficient

$$R \approx \frac{4}{3^{5/2}\pi^2} \frac{h}{\mu_{\mathrm{red}} m_{\mathrm{H}} \mathcal{Z}_i \mathcal{Z}_j e^2} S(E_0)\tau^2 e^{-\tau},$$

where

$$\tau = \frac{3E_0}{k_B T} = 42.46 \left[\mathcal{Z}_i^2 \mathcal{Z}_j^2 \mu_{\mathrm{red}} \left(\frac{T}{10^6 \ \mathrm{K}} \right)^{-1} \right]^{1/3}.$$

The corresponding rate at which the reaction generates energy per unit mass

$$q = \frac{\rho}{m_{\mathrm{H}}^2} \left(\frac{1}{1+\delta_{ij}} \right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} RQ.$$

Finally, the we can approximate the reaction coefficient at some particular temperature T as a powerlaw

$$R \approx R_0 T^\nu,$$

where the index $\nu = (\tau - 2)/3$. The energy generation rate can similarly be written $q = q_0 \rho T^\nu$, where the factor of ρ appears because the reaction rate varies as rate $\propto \rho^2 R$, and the energy per unit mass varies as rate$/\rho$.

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II. Major Fusion Reactions

With this framework established, we can apply it to the major fusion reactions of importance in stars. A quick note on notation: to help us keep track of charges, we will write symbols for nuclei as $^{\mathcal{A}}_{\mathcal{Z}}\text{C}$, where $\mathcal{A}$ is the atomic weight, $\mathcal{Z}$ is the atomic number, and C is the symbol for that element. This is slightly redundant, since a chemical symbol C uniquely identifies the atomic number $\mathcal{Z}$. However, writing out the numbers explicitly makes it easier to keep track of the charges, and to assure ourselves that they balance on each side of a reaction.

A. The $p - p$ Chain

1. Reaction Path

The most important mechanism for generating power in the Sun is known as the $p - p$ chain, for proton proton chain. It is not surprising that the reaction involves protons, i.e. hydrogen nuclei. These are by far the most abundance nuclei in main sequence stars, and, since the strength of the Coulomb barrier scales as $\mathcal{Z}_i \mathcal{Z}_j$, it is also the reaction with the lowest Coulomb barrier. Thus it begins at the lowest temperature.

[Slide 1 – binding energy per nucleon]

Before going into the details of the reaction, it is useful to re-examine the chart of binding energy per nucleon. Clearly the first big peak is at helium-4, so that is where we expect the reaction to go. However, getting there is not so easy, because all the stable nuclei shown in the chart except ^1_1H contain neutrons. The reason is that neutrons are required to provide enough nuclear force to hold a nucleus together against the Coulomb repulsion of the protons. Thus the most obvious first step for a reaction involving two hydrogen nuclei doesn't work. We can't easily do

$$^1_1\text{H} +^1_1\text{H} \ \rightarrow \ ^2_2\text{He}$$
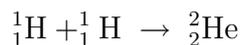
because ^2_2He is not a stable nucleus. Any ^2_2He made in such a manner almost immediately disintegrates into two protons, producing no net energy release.

Thus for a reaction to generate energy, one must find a way to bypass ^2_2He and jump to a stable state. One possible solution to this problem was discovered by Hans Bethe in 1939: during the very brief period that ^2_2He lives, a weak nuclear reaction can occur that converts one of the protons into a neutron plus a positron plus a neutrino. The positron and neutrino, which do not feel the strong nuclear force, immediately escape from the nucleus, leaving behind a proton plus a neutron. The proton plus neutron do constitute a stable nucleus: deuterium. The net reaction is exothermic, and the excess energy mostly goes into the recoil of the deuterium and positron. This excess energy is then turned into heat when the nuclei collide with other particle. The final reaction is

$$^1_1\text{H} + \ ^1_1\text{H} \ \rightarrow \ ^2_1\text{D} + e^+ + \nu_e$$

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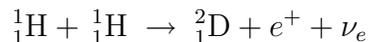
The electron neutrino, ν_e, escapes the star immediately, while the positron very quickly encounters an electron and annihilates, producing gamma rays which are then absorbed and converted into heat:

$$e^+ + e^- \rightarrow 2\gamma,$$
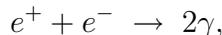
where γ is the symbol for photon. As we'll discuss further in a moment, the proton-neutron conversion is very unlikely because it relies on the weak force, so the reaction coefficient for this reaction is very small compared to others in the chain. In terms of our earlier notation, $S(E_0)$ is very small for this reaction.

The next step in the chain is an encounter between the deuterium nucleus and another proton, producing helium:
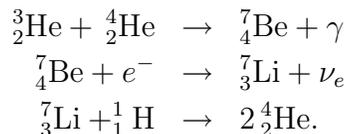
$$^2_1\text{D} + {}^1_1\text{H} \rightarrow {}^3_2\text{He} + \gamma.$$
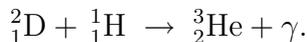
This reaction goes very quickly compared to the first step, because the Coulomb barrier is the same (deuterium and ordinary hydrogen both have $\mathcal{Z} = 1$), but no weak nuclear forces are required.

For the last part of the chain, there are three possibilities, known as the pp I, pp II, and pp III branches. Branch I involves an encounter between two ^3_2He nuclei produced in the previous step:

$$^3_2\text{He} + {}^3_2\text{He} \rightarrow {}^4_2\text{He} + 2\,{}^1_1\text{H}.$$
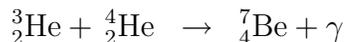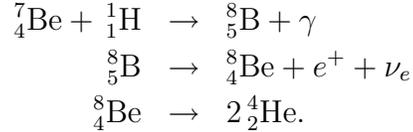
This reaction has a Coulomb barrier four times higher than the first one, but, since it does not require a weak nuclear interaction, it actually proceeds faster than the first step. At this point the reaction stops, because ^4_2He is stable, and there is no route from there to a more massive nucleus that is accessible at the temperatures of $\sim 10^7$ K where hydrogen burning occurs.

Branch II involves an encounter between the ^3_2He and a pre-existing ^4_2He nucleus to make beryllium, followed by capture of an electron to convert the beryllium to lithium, followed by capture of one more proton and fission of the resulting nucleus:

$$\begin{aligned}
^3_2\text{He} + {}^4_2\text{He} &\rightarrow {}^7_4\text{Be} + \gamma \\
^7_4\text{Be} + e^- &\rightarrow {}^7_3\text{Li} + \nu_e \\
^7_3\text{Li} + {}^1_1\text{H} &\rightarrow 2\,{}^4_2\text{He}.
\end{aligned}$$

Finally, branch III involves production of beryllium-7 just like the first step of the pp II branch, but then an encounter between that and another proton to produce boron. The boron then decays to beryllium via positron emission, and finally ends at beryllium-8, which spontaneously splits:

$$^3_2\text{He} + {}^4_2\text{He} \rightarrow {}^7_4\text{Be} + \gamma$$

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$$\begin{aligned}
{}^{7}_{4}\text{Be} + {}^{1}_{1}\text{H} &\rightarrow {}^{8}_{5}\text{B} + \gamma \\
{}^{8}_{5}\text{B} &\rightarrow {}^{8}_{4}\text{Be} + e^{+} + \nu_e \\
{}^{8}_{4}\text{Be} &\rightarrow 2\,{}^{4}_{2}\text{He}.
\end{aligned}$$

As before, the positron produced in the third step immediately encounters an electron and annihilates into gamma rays.

Which of these chains is most important depends on the local density, temperature, and chemical composition. Obviously pp II and pp III are more likely when there is more ${}^{4}_{2}\text{He}$ around, since they require it. In Sun, pp I is 69% of all reactions, pp II is 31%, and pp III is 0.1%.

2. Energetics and Rates

   The net reactions associated with these chains can be written:

   $$4\,{}^{1}_{1}\text{H} \rightarrow {}^{4}_{2}\text{He} + \text{photons, neutrinos, and light particles,}$$

   where the exact number of photons, neutrinos, and light particles depends on which branch is taken. The total energy release is given by subtracting the mass of He-4 from the mass of 4 protons, and is given by

   $$\Delta E = (4m_{\text{H}} - m_{\text{He}})c^2 = 26.73 \text{ MeV.}$$

   The actual amount of energy that goes into heating up the gas depends on the amount of energy carried away by neutrinos, which escape the star. This is different for each branch, because each branch involves production of a different number of neutrinos with different energies. The neutrino loss is 2.0% for pp I, 4.0% for pp II, and 28.3% for pp III.

   In any of the pp branch, the first step, which requires spontaneous conversion of a proton into a neutron, is by far the slowest, and thus the rate at which it occurs controls the rate for the entire chain. For this reason, we can calculate the rate coefficient simply by knowing the properties of this reaction. The reaction begins to occur at an ignition temperature that is roughly equal $T_i = 4 \times 10^6$ K. The Sun's central temperature $T_0 \approx 1.57 \times 10^7$ K, which gives

   $$\tau = \frac{3E_0}{k_B T} = 42.46 \left[\mathcal{Z}_i^2 \mathcal{Z}_j^2 \mu_{\text{red}} \left(\frac{T}{10^6 \text{ K}} \right)^{-1} \right]^{1/3} = 13.5.$$

   The reaction rate varies as temperature to roughly the 4th power. Measuring the value of $S(E_0)$ for this reaction lets us compute the rate coefficient. If we do not make the powerlaw approximation and just plug into

   $$R \approx \frac{4}{3^{5/2}\pi^2} \frac{h}{\mu_{\text{red}} m_{\text{H}} \mathcal{Z}_i \mathcal{Z}_j e^2} S(E_0) \tau^2 e^{-\tau},$$

   we get

   $$R \approx 6.34 \times 10^{-37} \left(\frac{T}{10^6 \text{ K}} \right)^{-2/3} \exp\left[-\frac{33.8}{(T/10^6 \text{ K})^{1/3}} \right] \text{ cm}^3/\text{ s.}$$

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If we multiply this by the number density of protons, we get an estimate for the rate of reactions that a single proton undergoes. The inverse of this is the lifetime of a proton – the amount of time it takes for it to react with another proton and begin the reaction chain that will turn it into helium. Thus the proton lifetime is

$$t = \frac{1}{n_p R} = \frac{m_{\mathrm{H}}}{\rho X R} = \frac{8.3 \times 10^4 \text{ yr}}{X} \left(\frac{1 \text{ g cm}^{-3}}{\rho} \right) \left(\frac{T}{10^6 \text{ K}} \right)^{2/3} \exp \left[\frac{33.8}{(T/10^6 \text{ K})^{1/3}} \right].$$

Plugging in a density of 100 g cm^{-3} and a temperature of 1.5×10^7 K, the result is a bit over 10^9 yr. Thus the typical proton in the center of the Sun requires $> 10^9$ yr to undergo fusion. Averaging over a larger volume of the Sun, which has a lower density and temperature, makes the timescale even longer.

Finally, combining the reaction rate coefficient R with an energy release of $Q = 13.4$ MeV per reaction (since 26.73 MeV is what we get when we use 4 protons, and each pp reaction only uses 2), the corresponding energy generation rate is

$$
\begin{aligned}
q &\approx \frac{\rho}{m_{\mathrm{H}}^2} \left(\frac{1}{1 + \delta_{ij}} \right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R Q \\
&= 2.4 \times 10^6 X^2 \left(\frac{\rho}{1 \text{ g cm}^{-3}} \right) \left(\frac{T}{10^6 \text{ K}} \right)^{-2/3} \exp \left[-\frac{33.8}{(T/10^6 \text{ K})^{1/3}} \right] \text{ erg g}^{-1} \text{ s}^{-1}.
\end{aligned}
$$

If we do want to make a powerlaw fit, the index is
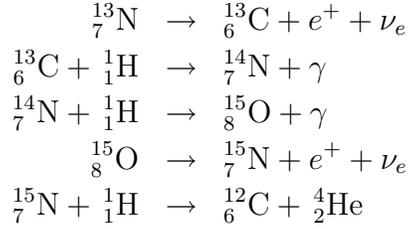
$$\nu = \frac{\tau - 2}{3} \approx 4.$$

B. The CNO Cycle
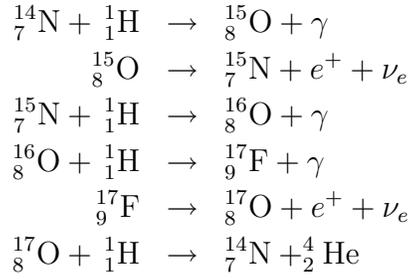
1. Reaction Path

The $p - p$ chain faces a relatively small Coulomb barrier, since the rate-limiting step has $\mathcal{Z} = 1$ for both reactants. However, it is slow because it requires spontaneous proton-neutron conversion within the short time that two protons are close to one another in a violently unstable configuration. There is another possible route to turning hydrogen into helium-4 which has a different tradeoff: a larger Coulomb barrier, but no need for a weak reaction in a short period.

This second route is called the CNO cycle, and was discovered independently by Hans Bethe and Carl-Friedrich von Weizsäcker in 1938. It relies on the fact that cabron, nitrogen, and oxygen are fairly abundant in the universe, and are present in a star even before it starts nuclear burning. They can act as catalysts in a proton fusion reaction. The reaction chain is

$$^{12}_{6}\mathrm{C} + ^{1}_{1}\mathrm{H} \quad \rightarrow \quad ^{13}_{7}\mathrm{N} + \gamma$$

$$\begin{aligned}
{}^{13}_{7}\mathrm{N} &\rightarrow {}^{13}_{6}\mathrm{C} + e^+ + \nu_e \\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + \gamma \\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + \gamma \\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + e^+ + \nu_e \\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}
\end{aligned}$$

Alternately, the chain can be:

$$\begin{aligned}
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + \gamma \\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + e^+ + \nu_e \\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{16}_{8}\mathrm{O} + \gamma \\
{}^{16}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{17}_{9}\mathrm{F} + \gamma \\
{}^{17}_{9}\mathrm{F} &\rightarrow {}^{17}_{8}\mathrm{O} + e^+ + \nu_e \\
{}^{17}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + {}^{4}_{2}\mathrm{He}
\end{aligned}$$

The first route is generally the more important one, by a large factor.

2. Energetics and Rates

Note that both of these chains have the property that it neither creates nor destroys any carbon or nitrogen nuclei. One starts with ${}^{12}_{6}\mathrm{C}$ and ends with it, or starts with ${}^{14}_{7}\mathrm{N}$ and ends with it. Thus the net reaction is exactly the same as for the $p-p$ chain:

$$4\,{}^{1}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + \text{photons, neutrinos, and light particles,}$$

In this sense, the carbon or nitrogen acts as a catalyst. They enable the reaction to take place, but are not themselves consumed or created by it. Since the net reaction is the same as for $p-p$, the net energy release is also the same, except for slightly different neutrino losses. For the CNO cycle, $Q \approx 25$ MeV once the neutrino losses are factored in, as opposed to 27 MeV for $p-p$.

In each of these reaction chains, it makes sense to distinguish between reactions that involve creation of a positron e^+ and reactions that do not. The former are called β decays, and they rely on the weak nuclear force. However, they are much faster than the first step of the $p-p$ chain, because the take place in nuclei that are stable except for the weak reaction they undergo. Thus there is no need for precisely timing the reaction with the period when two protons are in close proximity.

At the temperature found in the Sun, however, the rate-limiting step is not the β decays, but the need to overcome strong Coulomb barriers. The ignition temperature is about 1.5×10^7 K, about the Sun's central temperature. Analysis of the full reaction rate is tricky because which step is the rate-limiting one depends on the relative abundances of C, N, O, and the other catalysts,

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which are in turn determined by the reaction cycle itself. Once things reach equilibrium, however, it turns out that the step

$$^{14}_{7}\text{N} + ^{1}_{1}\text{H} \rightarrow ^{15}_{8}\text{O} + \gamma$$
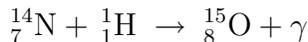
is the rate-limiting one. This step appears in both cycles.

Plugging $\mathcal{Z}_i = 7$, $\mathcal{Z}_j = 1$, and $\mu_{\text{red}} = (14)(1)/(14+1) = 0.93$ into our equation for the temperature-dependence gives

$$\tau = \frac{3E_0}{k_B T} = 42.46 \left[\mathcal{Z}_i^2 \mathcal{Z}_j^2 \mu_{\text{red}} \left(\frac{T}{10^6 \text{ K}} \right)^{-1} \right]^{1/3} = 152 \left(\frac{T}{10^6 \text{ K}} \right)^{-1/3}$$

Using the laboratory measurement for $S(E_0)$ for this reaction, the rate coefficient is

$$R = 8.6 \times 10^{-19} \left(\frac{T}{10^6 \text{ K}} \right)^{-2/3} \exp \left[-\frac{152}{(T/10^6 \text{ K})^{1/3}} \right] \text{ cm}^3 / \text{ s},$$

and the corresponding energy generation rate is

$$q = 8.7 \times 10^{27} X X_{\text{CNO}} \left(\frac{\rho}{1 \text{ g cm}^{-3}} \right) \left(\frac{T}{10^6 \text{ K}} \right)^{-2/3} \exp \left[-\frac{152}{(T/10^6 \text{ K})^{1/3}} \right] \text{ erg g}^{-1} \text{ s}^{-1},$$

where X_{CNO} is the total mass fraction of carbon, nitrogen, and oxygen. This is roughly $Z/2$, where Z is the total mass fraction of metals.

It is informative to evaluate q for the $p-p$ chain and for the CNO cycle using values appropriate to the center of the Sun: $\rho \approx 100$ g cm^{-3}, $T \approx 1.5 \times 10^7$ K, $X = 0.71$, $Z = 0.02$. This gives

$$q_{p-p} = 82 \text{ erg g}^{-1} \text{ s}^{-1}$$
$$q_{\text{CNO}} = 6.4 \text{ erg g}^{-1} \text{ s}^{-1}$$

Thus the $p-p$ chain dominates in the Sun by about a factor of 10. However, it is important to notice that, because it has 152 instead of 33.8 in the exponential, the CNO cycle is much more temperature-sensitive than the $p-p$ chain. If we assign a powerlaw approximation, the index is

$$\nu = \frac{\tau - 2}{3} = 20.$$

Thus stars a bit more massive than the Sun, which we will see have higher central temperatures, the CNO cycle dominates. In stars smaller than the Sun, the CNO cycle is completely irrelevant.

This also brings out a general feature of all the nuclear reactions we will consider: the temperature-sensitivity is determined by τ, and τ in turn depends on the charges of the nuclei involved, $\mathcal{Z}$, because it is determined

by the Coulomb barrier. The stronger the nuclear charge, the stronger the Coulomb barrier, and thus the higher the ignition temperature and the more temperature-sensitive the reaction becomes. We have already seen that the CNO cycle produces energy as a rate that varies as T^{20}, and the temperature-sensitivity only gets stronger as we march up the periodic table.

C. The Triple-α Process

1. Reaction Path

The transition from hydrogen to helium-4 takes us to the first big peak in the binding energy per nucleon curve, and moves us from a reaction where the Coulomb barrier to the first step is 1 ($\mathcal{Z}_i = \mathcal{Z}_j = 1$) to one where it is 4 ($\mathcal{Z}_i = \mathcal{Z}_j = 2$). In fact, we'll see in a moment that it is even worse than that. As a result, the temperature required to burn any further up the periodic table is significantly higher than that required to burn hydrogen. Hydrogen burning tends to keep stars' central temperatures about constant while hydrogen remains available, so helium burning is generally not initiated until all the hydrogen has been exhausted, which means that the star must leave the main sequence. Thus all the reactions we will talk about for the remainder of the class take place in post-main-sequence stars.
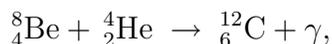
[Slide 1 – binding energy per nucleon]

In the core of such a star, essentially all the hydrogen will have been converted to helium-4. Looking at the curve of binding energy per nucleon, the next peak is clearly carbon-12. However, we again see a problem in getting there. The most obvious reaction to start is

$$\begin{smallmatrix}4\\2\end{smallmatrix}\text{He} + \begin{smallmatrix}4\\2\end{smallmatrix}\text{He} \leftrightarrows \begin{smallmatrix}8\\4\end{smallmatrix}\text{Be}.$$

However, the $\begin{smallmatrix}8\\4\end{smallmatrix}$Be nucleus is violently unstable, and disintegrates in about 3×10^{-16} s. Nor can we get out of the problem by hoping for a weak reaction to convert a proton into a neutron, because there is no stable nucleus with an atomic mass number $\mathcal{A} = 8$.

Thus we need to jump past atomic number 8 in order to burn He. The solution to this problem was found by Edwin Salpeter in 1952. If the density and temperature get high enough, it may be possible for the $\begin{smallmatrix}8\\4\end{smallmatrix}$Be nucleus to collide with another $\begin{smallmatrix}4\\2\end{smallmatrix}$He nucleus before it decays. Then it will undergo the reaction

$$\begin{smallmatrix}8\\4\end{smallmatrix}\text{Be} + \begin{smallmatrix}4\\2\end{smallmatrix}\text{He} \rightarrow \begin{smallmatrix}12\\6\end{smallmatrix}\text{C} + \gamma,$$

and arrive at carbon-12, which is stable and is another peak of binding energy per nucleon. This reaction is known as the triple-α process, because it effectively involves a three-way collision between three helium-4 nuclei, which are also known as α particles. It is not a true three-way collision, because some extra time for the third collision is provided by the lifetime of the $\begin{smallmatrix}8\\4\end{smallmatrix}$Be nucleus, but it is nearly so.

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In addition to the short-lived beryllium state, another factor that helps this reaction go is the existence of a resonance in the carbon-12 nucleus that coincides closely in energy with that produced by colliding another helium nucleus with beryllium-8. This greatly enhances the rate at which the second step in the reaction chain takes place.

In environments where a significant amount of carbon builds up and the temperature is high, carbon will occasionally capture an additional helium nucleus and jump to the next peak in the binding energy curve, oxygen-16:

$$\ce{^{12}_{6}C} + \ce{^{4}_{2}He} \rightarrow \ce{^{16}_{8}O} + \gamma.$$
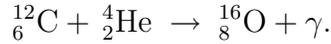
Thus stars in which the triple-α process takes place wind up containing a mixture of carbon and oxygen, with the exact ratio depending on their age, density, and temperature. Further He captures are also possible, but become increasingly unlikely as one moves up in atomic number due to the increasing Coulomb barrier.

2. Energetics and Rates

The first step in the triple-α process is actually endothermic, although only mildly so. The mildly endothermic nature of the reaction is important. The fact that it is endothermic is the reason that the $\ce{^{8}_{4}Be}$ nucleus is unstable: it can spontaneously split back into two helium nuclei. That is is only mildly endothermic (it requires 92 keV) means that collision between nuclei moving around at the thermal speed are always producing some of it, so there is always a small amount of beryllium-8 present.

The second step, converting beryllium-8 to carbon, is exothermic. The net energy released can be calculated by comparing the mass of the carbon-12 nucleus to that of three helium-4 nuclei:

$$Q = (3m_{\mathrm{He}} - m_{\mathrm{C}})c^2 = 7.28 \text{ MeV}.$$

The capture of a fourth He nucleus leading to oxygen-16 yields another 7.16 MeV.

To compute the reaction rate and its temperature-dependence, one can assume that there is always a small amount of $\ce{^{8}_{4}Be}$ by equating the creation and destruction rates – a process that we will not go through, but which yields an amount of $\ce{^{8}_{4}Be}$ that is roughly independent of temperature. It does not depend on temperature because the limiting factor in how much beryllium is present is the very rapid spontaneous decay of the beryllium nucleus, not the Coulomb barrier to creating it. Calculations show that the beryllium fraction is ~ 1 part in 10^{10}.

All the temperature-dependence in the reaction rate comes in the next step that of converting beryllium-8 to carbon-12. As discussed a moment ago, the reaction process for creating carbon-12 depends on a resonance. We will not

go through the details of how to calculate a resonant reaction in class, but we can sketch it out briefly in order to understand the temperature-dependence of the reaction. Recall from last class that the reaction rate is proportional to

$$R \propto \int_0^\infty \sigma(E) E e^{-E/k_B T} \, dE.$$

For a non-resonant reaction, we evaluated this by using a calculation of quantum tunneling to estimate $\sigma(E)$. For a resonant reaction, however, the process is much simpler: when there is a dominant resonance, essentially all reactions take place at energies very close to the energy required to hit the resonance. For this reason we can treat the factor $E e^{-E/k_B T}$ as nearly constant over the resonance, and take it out of the integral, yielding

$$R \propto E_R e^{-E_R/k_B T} \int_0^\infty \sigma(E) \, dE,$$

where E_R is the energy that the incoming particle must have in order to hit the resonance. Then if we let

$$\tau_R = \frac{E_R}{k_B T},$$

we have

$$R \propto e^{-\tau_R} \int_0^\infty \sigma(E) \, dE.$$

As with the non-resonant case, all the temperature-dependence is encapsulated in the parameter τ_R, which varies as T^{-1}.

The second step in the triple-α process relies on a resonance that is at an energy $E_R = 379.5$ keV above the energy of the beryllium-8 nucleus, so that is the energy an incoming particle must have to trigger the resonance. (Note that the state in question has an energy 7.95 MeV above the ground state of carbon-12, but the relevant question is the difference between that energy and the energy of the beryllium-8 nucleus, which is much smaller.)

$$\tau_R = \frac{379.5 \, \text{keV}}{k_B T} = 44.0 \left(\frac{10^8 \, \text{K}}{T} \right).$$

This is normalized to 10^8 K, which is about the ignition temperature for this reaction.

To go further in computing the reaction rate, we must recall that triple-α effectively requires a three-way collision. For a single particle, we said that the rate at which it encounters other particles is proportional to n. We can also view this as a probability: the probability of a collision per unit time is proportional to n. For a three-body process, we need to ask about the probability of two of them striking simultaneously or nearly so (within the

10

10^{-16} s lifetime of the $^8_4$Be nucleus. The rate at which such double-collisions occurs is proportional to the probability of one collision times the probability of another: $n^2 v^2$. Thus we expect a collision rate that varies as n^2. We will not walk through putting this in terms of a rate coefficient, but a straightforward generalization of our existing calculation shows that the reaction rate per unit volume varies as $R n^3$, while the kinetic part rate coefficient itself varies as T^{-3} – it is T^{-3} instead of the usual $T^{-3/2}$ because the collision rate varies as n^2 rather than n.

Putting this together, we expect the rate coefficient to vary with temperature as

$$R \propto T^{-3} \int_0^\infty \sigma(E) E e^{-E/k_B T}\, dE \propto \tau_R^3 e^{-\tau_R} \int_0^\infty \sigma(E)\, dE.$$

It will vary with density as ρ^2. Putting in the measured cross sections, the final result for the energy generation rate is

$$q = 5.1 \times 10^8 Y^3 \left(\frac{\rho}{1\,\mathrm{g\ cm^{-3}}}\right)^2 \left(\frac{T}{10^8\,\mathrm{K}}\right)^{-3} \exp\left(-\frac{44}{T/10^8\,\mathrm{K}}\right)\ \mathrm{erg\ g^{-1}\ s^{-1}}.$$
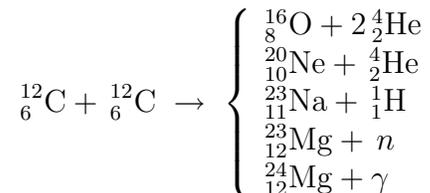
To get a powerlaw fit for the temperature-dependence, we can proceed exactly as we did in the non-resonant case. The slope in a log-log plot of R vs. T is

$$
\begin{aligned}
\nu = \frac{d\ln R}{d\ln T} &= \frac{d}{d\ln T}\left(3\ln\tau_R - \tau_R + \mathrm{const}\right)\\
&= \frac{d}{d\ln T}\left[-3\ln T - \tau_{R,0}\left(\frac{T}{T_0}\right)^{-1}\right]\\
&= -3 - \tau_{R,0} T_0 T \frac{d}{dT}\left(\frac{1}{T}\right)\\
&= -3 + \frac{\tau_{R,0} T_0}{T}\\
&= \tau_R - 3\\
&\approx 41
\end{aligned}
$$

Thus we see that the triple-α reaction is extraordinarily temperature-sensitive.

D. Carbon and Oxygen Burning

At even higher temperatures, the Coulomb barrier for oxygen and carbon can be overcome, creating yet heavier nuclei. Many reaction paths are possible. Carbon can burn to produce

$$^{12}_6\mathrm{C} + {}^{12}_6\mathrm{C} \rightarrow \begin{cases} {}^{16}_8\mathrm{O} + 2\,{}^4_2\mathrm{He}\\ {}^{20}_{10}\mathrm{Ne} + {}^4_2\mathrm{He}\\ {}^{23}_{11}\mathrm{Na} + {}^1_1\mathrm{H}\\ {}^{23}_{12}\mathrm{Mg} + n\\ {}^{24}_{12}\mathrm{Mg} + \gamma \end{cases}$$

Oxygen burning produces similar reactions, which I will not write down – they are in your textbook. The final outcome of this sort of burning is generally $^{32}_{16}$Si, silicon.

These reactions begin at temperatures of about 6×10^8 K. Given the number of possible pathways and reactions involved, there isn't a single simple formula for the energy generation rate. Calculations of reactions of this sort are generally done by a computer, which keeps track of the densities of many different nuclei and calculates all the possible reactions and the energies they yield.
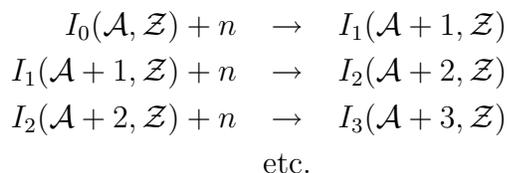
E. Silicon to Iron

At still higher temperatures, around 3×10^9 K, the typical photon becomes energetic enough that it can disrupt nuclei, knocking pieces off them in a process known as photodisintegration. The chemical balance in the star is then determined by a competition between this process and reactions between nuclei. However, as we might expect, the net effect is to drive the chemical balance ever further toward the most stable nucleus, iron. Once the temperature around 3×10^9 K, more and more nuclei begin to convert to $^{56}_{26}$Fe, and its close neighbors cobalt and nickel. Things stay in this state until the temperature is greater than about 7×10^9 K, at which point photons have enough energy to destroy even iron, and the entire process reverses: all elements are converted back into its constituent protons and neutrons, and photons reign supreme.

III. The r and s Processes

We have already seen how elements up through iron are built, but we have not yet mentioned how even heavier elements can be created. The answer is that they are not made in stars under normal circumstances, because when the only forces at work at electromagnetism and nuclear forces, it is never energetically favorable to create such elements in any significant number. Creating such elements requires the intervention of another force: gravity.

When stars are in the process of being crushed by gravity, right before they explode as supernovae (which we will discuss toward the end of the course), gravity drives a process that converts most of the protons to neutrons. This creates a neutron-rich environment unlike any found at earlier stages of stellar evolution, when the lack of neutrons was often the rate-limiting step.

In a neutron-rich environment, it becomes possible to create heavy nuclei via the absorption of neutrons by existing nuclei. Since the neutrons are neutral, there is no Coulomb barrier to overcome, and the reaction proceeds as quickly as the neutron supply allows. Reactions look like this:

$$
\begin{aligned}
I_0(\mathcal{A}, \mathcal{Z}) + n &\rightarrow I_1(\mathcal{A} + 1, \mathcal{Z}) \\
I_1(\mathcal{A} + 1, \mathcal{Z}) + n &\rightarrow I_2(\mathcal{A} + 2, \mathcal{Z}) \\
I_2(\mathcal{A} + 2, \mathcal{Z}) + n &\rightarrow I_3(\mathcal{A} + 3, \mathcal{Z})
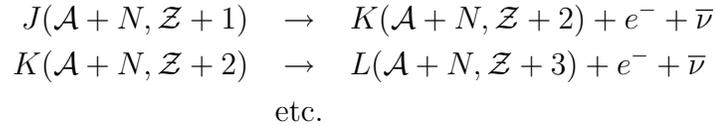\end{aligned}
$$

etc.

Neutron captures therefore increase $\mathcal{A}$ at constant $\mathcal{Z}$.

This continues until it produces a nucleus that is unstable and undergoes β decay, converting one of the neutrons back into a proton:

$$I_N(\mathcal{A} + N, \mathcal{Z}) \;\rightarrow\; J(\mathcal{A} + N, \mathcal{Z} + 1) + e^- + \overline{\nu}.$$

(The bar over the ν indicates that this is an anti-neutrino.) If the new element produced in this way is stable, it will begin neutron capturing again. If not, it will keep undergoing β decays until it becomes stable:

$$\begin{aligned}
J(\mathcal{A} + N, \mathcal{Z} + 1) &\;\rightarrow\; K(\mathcal{A} + N, \mathcal{Z} + 2) + e^- + \overline{\nu} \\
K(\mathcal{A} + N, \mathcal{Z} + 2) &\;\rightarrow\; L(\mathcal{A} + N, \mathcal{Z} + 3) + e^- + \overline{\nu} \\
&\text{etc.}
\end{aligned}$$

β decays therefore increase $\mathcal{Z}$ at constant $\mathcal{A}$.

These processes together lead to the build-up of elements heavier than iron. The chain stops if at any point it reaches a nucleus that is stable against β decay, and is also not able to capture neutrons because neutron capture is endothermic for it, i.e. adding a neutron makes the nucleus less rather than more bound.

The rate of neutron capture depends on the local density and temperature, while the rate of β decay does not. As a result, either can be the rate-limiting step in the build-up, depending on the local environment and the element in question. Elements that are build up by reaction chains in which β decays occur faster are called r process, for rapid. Elements where β decays are slower are called s process, for slow. Knowing which process produces which elements requires knowing the stability, binding energy, and β decay lifetimes of the various elements, which must be determined experimentally.