

Class 6 Notes: Internal Energy and Radiative Transfer

In the last class we used the kinetic theory of gasses to understand the pressure of stellar material. The kinetic view is essential to generalizing the concept of pressure to the environments found in stars, where gas can be relativistic, degenerate, or both. The goal of today's class is to extend that kinetic picture by thinking about pressure in stars in terms of the associated energy content of the gas. This will let us understand how the energy flow in stars interacts with gas pressure, a crucial step toward building stellar models.

I. Pressure and Energy

A. The Relationship Between Pressure and Internal Energy

The fundamental object we dealt with in the last class was the distribution of particle momenta, $dn(p)/dp$, which we calculated from the Boltzmann distribution. Given this, we could compute the pressure. However, this distribution also corresponds to a specific energy content, since for particles that don't have internal energy states, internal energy is just the kinetic energy of particle motions. Given a distribution of particle momenta $dn(p)/dp$ in some volume of space, the corresponding density of energy within that volume of space is

$$e = \int_0^\infty \frac{dn(p)}{dp} \epsilon(p) dp,$$

where $\epsilon(p)$ is the energy of a particle with momentum p . It is often more convenient to think about the energy per unit mass than the energy per unit volume. The energy per unit mass is just e divided by the density:

$$u = \frac{1}{\rho} \int_0^\infty \frac{dn(p)}{dp} \epsilon(p) dp.$$

The kinetic energy of a particle with momentum p and rest mass m is

$$\epsilon(p) = mc^2 \left(\sqrt{1 + \frac{p^2}{m^2c^2}} - 1 \right).$$

This formula applies regardless of p . In the limit $p \ll mc$ (i.e. the non-relativistic case), we can Taylor expand the square root term to $1 + p^2/(2m^2c^2)$, and we recover the usual kinetic energy: $\epsilon(p) = p^2/(2m)$. In the limit $p \gg mc$ (the ultra-relativistic case), we can drop the plus 1 and the minus 1, and we get $\epsilon(p) = pc$.

Plugging the non-relativistic, non-degenerate values for $\epsilon(p)$ and $dn(p)/dp$ into the integral for u and evaluating gives

$$u = \frac{1}{\rho} \int_0^\infty \frac{4\pi n}{(2\pi m k_B T)^{3/2}} p^2 e^{-p^2/(2mk_B T)} \left(\frac{p^2}{2m} \right) dp$$

$$\begin{aligned}
&= \frac{2\pi}{m^2(2\pi mk_B T)^{3/2}} \int_0^\infty p^4 e^{-p^2/(2mk_B T)} dp \\
&= \frac{4}{m\pi^{1/2}} k_B T \int_0^\infty q^4 e^{-q^2} dq \\
&= \frac{3}{2m} k_B T \\
&= \frac{3P}{2\rho},
\end{aligned}$$

where the integral over q evaluates to $3\sqrt{\pi}/8$. This is the same as the classic result that an ideal gas has an energy per particle of $(3/2)k_B T$. Since the pressure and energies are simply additive, it is clear that the result $u = (3/2)(P/\rho)$ applies even when there are multiple species present.

Applying the same procedure in the relativistic, non-degenerate limit gives

$$\begin{aligned}
u &= \frac{1}{\rho} \int_0^\infty \left(\frac{c}{k_B T}\right)^3 \frac{n}{2} p^2 e^{-pc/k_B T} (pc) dp \\
&= \frac{c^4}{2m(k_B T)^3} \int_0^\infty p^3 e^{-pc/k_B T} dp \\
&= \frac{3}{m} k_B T \\
&= 3\frac{P}{\rho}.
\end{aligned}$$

It is straightforward to show that this result applies to radiation too, by plugging $\epsilon = h\nu$ for the energy and the Planck distribution for $dn(\nu)/d\nu$. Note that this implies that the volume energy density of a thermal radiation field is

$$e_{\text{rad}} = aT^4 = 3P_{\text{rad}}.$$

For the non-relativistic, degenerate limit we have a step-function distribution that has a constant value $dn(p)/dp = (8\pi/h^3)p^2$ out to some maximum momentum $p_0 = [3h^3 n/(8\pi)]^{1/3}$, so the energy is

$$\begin{aligned}
u &= \frac{1}{\rho} \int_0^{p_0} \frac{8\pi}{h^3} p^2 \left(\frac{p^2}{2m}\right) dp \\
&= \frac{4\pi}{5m\rho h^3} p_0^5 \\
&= \left(\frac{3}{\pi}\right)^{2/3} \frac{3h^2 n^{5/3}}{40\rho m} \\
&= \frac{3P}{2\rho},
\end{aligned}$$

exactly as in the non-degenerate case.

Finally, for the relativistic degenerate case we have

$$\begin{aligned}
 u &= \frac{1}{\rho} \int_0^{p_0} \frac{8\pi}{h^3} p^2 (pc) dp \\
 &= \left(\frac{3}{\pi}\right)^{1/3} \frac{3}{8} \frac{hcn^{4/3}}{\rho} \\
 &= 3 \frac{P}{\rho}
 \end{aligned}$$

B. Adiabatic Processes and the Adiabatic Index

Part of the reason that internal energies are interesting to compute is because of the problem of adiabatic processes. An adiabatic process is one in which the gas is not able to exchange heat with its environment or extract it from internal sources (like nuclear burning), so any work it does must be balanced by a change in its internal energy. The classic example of this is a gas that is sealed in an insulated box, which is then compressed or allowed to expand. In many circumstances we can think of most of the gas in a star (that outside the region where nuclear burning takes place) as adiabatic. It can exchange energy with its environment via radiation, but, as we have previously shown, the radiation time is long compared to the dynamical time. Thus any process that takes place on timescale shorter than a Kelvin-Helmholtz timescale can be thought of as adiabatic.

To understand how an adiabatic gas behaves, we use the first law of thermodynamics, which we derived a few classes back:

$$\frac{du}{dt} + P \frac{d}{dt} \left(\frac{1}{\rho} \right) = q - \frac{\partial F}{\partial m} = 0,$$

where we have set the right-hand side to zero under the assumption that the gas is adiabatic, so it does not exchange heat with its environment and does not generate heat by nuclear fusion. We have just shown that for many types of gas $u = \phi P/\rho$, where ϕ is a constant that depends on the type of gas. If we make this substitution in the first law of thermodynamics, then we get

$$0 = \phi P \frac{d}{dt} \left(\frac{1}{\rho} \right) + \phi \frac{1}{\rho} \frac{d}{dt} P + P \frac{d}{dt} \left(\frac{1}{\rho} \right) = (\phi + 1) P \frac{d}{dt} \left(\frac{1}{\rho} \right) + \phi \frac{1}{\rho} \frac{d}{dt} P$$

This implies that

$$\begin{aligned}
 \frac{dP}{dt} &= -\frac{\phi + 1}{\phi} \rho P \frac{d}{dt} \left(\frac{1}{\rho} \right) \\
 &= \left(\frac{\phi + 1}{\phi} \right) \frac{P}{\rho} \frac{d\rho}{dt} \\
 \frac{dP}{P} &= \left(\frac{\phi + 1}{\phi} \right) \frac{d\rho}{\rho} \\
 \ln P &= \gamma_a \ln \rho + \ln K_a \\
 P &= K_a \rho^{\gamma_a},
 \end{aligned}$$

where $\gamma_a = (\phi + 1)/\phi$ and K_a is a constant. Thus we have shown that, for an adiabatic gas, the pressure and density are related by a powerlaw.

The constant of integration K_a is called the adiabatic constant, and it is determined by the entropy of the gas. The exponent γ_a is called the adiabatic index, and it is a function solely of the type of gas: all monatomic ideal gasses have $\gamma_a = (3/2 + 1)/(3/2) = 5/3$, whether they are degenerate or not. All relativistic gasses have $\gamma_a = 4/3$, whether they are degenerate or not.

The adiabatic index is a very useful quantity to know for a gas, because it describes how strongly that gas resists being compressed – it specifies how rapidly the pressure rises in response to an increase in density. The larger the value of γ_a , the harder it is to compress a gas. In a few weeks, we will see that the value of the adiabatic index for material in a star has profound consequences for the star's structure. For most stars γ_a is close to $5/3$ because the gas within them is non-relativistic. However, as the gas becomes more relativistic, γ_a approaches $4/3$, and resistance to compression drops. When that happens the star is not long for this world. A similar effect is responsible for star formation. Under some circumstances, radiative effects cause interstellar gas clouds to act as if they had $\gamma_a = 1$, which means very weak resistance to compression. The result is that these clouds collapse, which is how new stars form.

C. The Adiabatic Index for Partially Ionized Gas

The adiabatic index is fairly straightforward for something like a pure degenerate or non-degenerate gas, but the idea can be generalized to considerably more complex gasses. One case that is of particular interest is a partially ionized gas. This will be the situation in the outer layers of a star, where the temperature falls from the mean temperature, where the gas is fully ionized, to the surface temperature, where it is fully neutral. This case is tricky because the number of free gas particles itself becomes a function of temperature, and because the potential energy associated with ionization and recombination becomes an extra energy source or sink for the gas.

Consider a gas of pure hydrogen within which the number density of neutral atoms is n_0 and the number densities of free protons and electrons are $n_p = n_e$. The number density of all atoms regardless of their ionization state is $n = n_e + n_0$. We define the ionization fraction as

$$x = \frac{n_e}{n},$$

i.e. x is the fraction of all the electrons present that are free, or, equivalently, the fraction of all the protons present that do not have attached electrons. The pressure in the gas depends on how many free particles there are:

$$P = n_e k_B T + n_p k_B T + n_0 k_B T = (1 + x) n k_B T = (1 + x) \mathcal{R} \rho T$$

Thus the pressure at fixed temperature is higher if the gas is more ionized, because there are more free particles.

The number densities of free electrons and protons are determined by the Saha equation, which we encountered in the first class. As a reminder, the Saha equation is that the number density of ions at ionization states i and $i + 1$ are related by

$$\frac{n_{i+1}}{n_i} = \frac{2Z_{i+1}}{n_e Z_i} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi/k_B T},$$

where χ is the ionization potential and Z_i and Z_{i+1} are the partition functions of the two states. Applying this equation to the neutral and ionized states of hydrogen gives

$$\frac{n_e^2}{n_0} = \frac{2}{Z_0} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi/k_B T},$$

where Z_0 is the partition function of the neutral hydrogen, and we have set $Z_1 = 1$ because the ionized hydrogen has a partition function of 1. Making the substitution $n_e = xn$ and $n_0 = (1 - x)n$, the equation becomes

$$\frac{x^2}{1 - x} n = \frac{2}{Z_0} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi/k_B T}.$$

Finally, if we use the pressure relation to write $n = P/[(1 + x)k_B T]$, we arrive at

$$\frac{x^2}{1 - x^2} = \left(\frac{2}{h^3 Z_0} \right) \frac{(2\pi m_e)^{3/2} (k_B T)^{5/2}}{P} e^{-\chi/k_B T}.$$

This equation gives us the ionization fraction in terms of P and T .

To compute the adiabatic index, we must first know the internal energy. For a partially ionized gas this has two components. The first is the standard kinetic one, $(3/2)(P/\rho)$. However, we must also consider the ionization energy: neutral atoms have a potential energy that is lower than that of ions by an amount $\chi = 13.6$ eV. Thus the total specific internal energy including both kinetic and potential parts is

$$u = \frac{3P}{2\rho} + \frac{\chi}{m_H} x.$$

The second term says that the potential energy per unit mass associated with ionization is 13.6 eV per hydrogen mass, multiplied by the fraction of hydrogen atoms that are ionized. If none are then this term is 0, and if they all are, then the energy is 13.6 eV divided by per hydrogen atom mass.

Now we plug into the first law of thermodynamics for an adiabatic gas, $du/dt + P(d/dt)(1/\rho) = 0$:

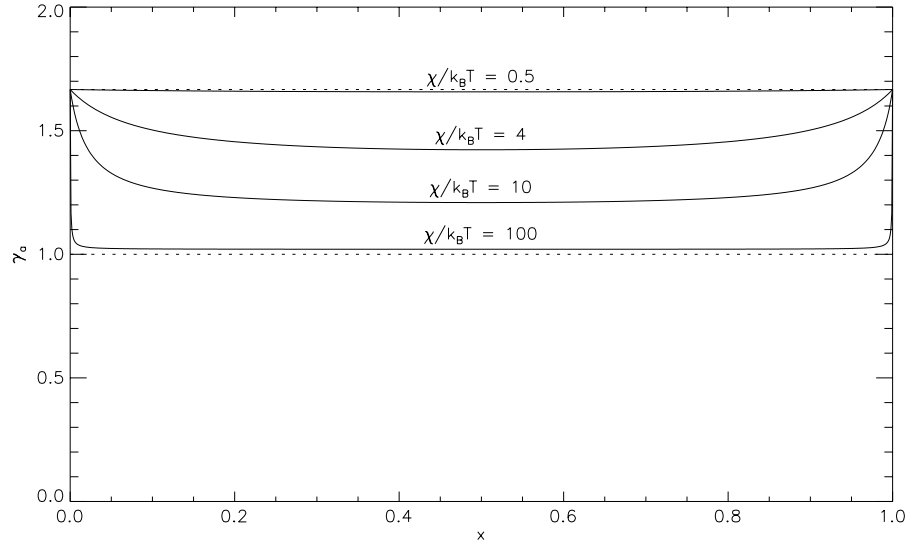
$$\begin{aligned} \frac{3}{2} \left(\frac{1}{\rho} \right) \frac{dP}{dt} - \frac{3P}{2\rho^2} \frac{d\rho}{dt} + \frac{\chi}{m_H} \frac{\partial x}{\partial \rho} \frac{d\rho}{dt} + \frac{\chi}{m_H} \frac{\partial x}{\partial P} \frac{dP}{dt} - \frac{P}{\rho^2} \frac{d\rho}{dt} &= 0 \\ \left[\frac{3}{2} + \chi n \left(\frac{\partial x}{\partial P} \right) \right] \frac{dP}{P} - \left[\frac{5}{2} - \frac{\chi n \rho}{P} \left(\frac{\partial x}{\partial \rho} \right) \right] \frac{d\rho}{\rho} &= 0 \\ \left[\frac{3}{2} + \frac{\chi}{k_B T} \left(\frac{P}{1+x} \right) \left(\frac{\partial x}{\partial P} \right) \right] \frac{dP}{P} - \left[\frac{5}{2} - \frac{\chi}{k_B T} \left(\frac{\rho}{1+x} \right) \left(\frac{\partial x}{\partial \rho} \right) \right] \frac{d\rho}{\rho} &= 0. \end{aligned}$$

In the second step we multiplied by ρ/P , and in the third step we substituted $n = P/[(1+x)k_B T]$, so that everything is in terms of x and T . Note that we have pulled a small trick in that we have ignored the temperature dependence of the partition function Z_0 . This is justified because it doesn't change much over the temperatures where ionization occurs.

The remainder of the calculation is straightforward but algebraically tedious. One evaluates the partial derivatives using the formula for x derived from the Saha equation, then integrates to get the dependence between P and ρ . The final result is

$$\gamma_a = \frac{5 + \left(\frac{5}{2} + \frac{\chi}{k_B T}\right)^2 x(1-x)}{3 + \left[\frac{3}{2} + \left(\frac{3}{2} + \frac{\chi}{k_B T}\right)^2\right] x(1-x)}.$$

This has the limiting behavior we would expect. For $x \rightarrow 0$ or $x \rightarrow 1$, the result approaches $5/3$, the value expected for a monatomic gas. In between γ_a is lower.



The amount by which γ_a drops depends on $\chi/k_B T$. When $\chi/k_B T$ is small, γ_a doesn't change much even when $x = 0.5$. When $\chi/k_B T$ is large, however, then for any x appreciably different from 0, the gas drops to near $\gamma_a = 1$.

We can understand this result intuitively. When $\chi/k_B T$ is small, ionizing an atom doesn't take much energy compared to the thermal energy, so the fact that there is an additional energy source or sink doesn't make much difference. When it is large, however, then every ionization requires a huge amount of thermal energy, and every recombination provides a huge amount of thermal energy. The value $\gamma_a = 1$ has a special significance: $P \propto \rho^1$ is what we expect for a gas that is isothermal, meaning at constant temperature. The reason that γ_a is close to 1 for a partially ionized gas where $\chi/k_B T$ is large is that any excess energy from compression is immediately used up in ionizing the gas just a little bit more, and any work done by the expansion of the gas is immediately balanced by reducing

the ionization state just a tiny bit, releasing a great deal of thermal energy. Thus partial ionization can, at certain temperatures, act like a thermostat that keeps the gas at fixed temperature as it expands or contracts.

This phenomenon in a different guise is familiar from every day life. When one heats water on a stove, the temperature rises to 100 C and the water begins to boil. While it is boiling, however, the water doesn't get any hotter. Its temperature stays at 100 C even as more and more heat is pumped into it. That is because all the extra heat is going into changing the phase of the water from liquid to gas, and the energy per molecule required to make that phase change happen is much larger than $k_B T$. The potential energy associated with the phase change causes the temperature to stay fixed until all the water boils away.

II. Radiative Transfer

Having understood how pressure and energy are related and what this implies for adiabatic gas, we now turn to the topic of how energy moves through stars. Although there are many possible mechanisms, the most ubiquitous is radiative transfer. The basic idea of radiative transfer is that the hot material inside a star reaches thermal equilibrium with its local radiation field. The hot gas cannot move, but the photons can diffuse through the gas. Thus when a slightly cooler fluid element in a star sits on top of a slightly hotter one below it, photons produced in the hot element leak into the cool one and heat it up. This is the basic idea of radiative transfer.

A. Opacity

In order to understand how radiation moves energy, we need to introduce the concepts of radiation intensity and matter opacity. First think about a beam of radiation. To describe the beam I need to specify how much energy it carries per unit area per unit time. I also need to specify its direction, giving the solid angle into which it is aimed. Finally, the beam may contain photons of many different frequencies, and I need to give you this information about each frequency. We define the object that contains all this information as the radiation intensity I , which has units of energy per unit time per unit area per unit solid angle per unit frequency. At any point in space, the radiation intensity is a function of direction and of frequency.

Before making use of this concept, it is important to distinguish between intensity and the flux of radiation that you're used to thinking about, H , which has units of energy per unit time per unit area per unit frequency. They are related very simply: flux is just the average of intensity over all directions. To understand the difference, imagine placing a sensor inside an oven whose walls are of uniform temperature. There is radiation coming from all directions equally, so the net flux of radiation is zero – as much energy moves from the left to the right each second as move from the right to the left. However, the intensity is not zero. The sensor would report that photons were striking it all the time, in equal numbers from every direction. Formally, we can define the relationship as follows. Suppose we

want to compute the flux in the z direction. This is given by the average of the intensity over direction:

$$H = \int I \cos \theta \, d\Omega = \int_0^{2\pi} \int_0^\pi I(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi.$$

Now consider aiming a beam traveling in some direction at a slab of gas. If the slab consists of partially transparent material, only some of the radiation will be absorbed. An example is shining a flashlight through misty air. The air is not fully opaque, but it is not fully transparent either, so the light is partially transmitted. The opacity of a material is a measure of its ability to absorb light. To make this formal, suppose the slab consists of material with density ρ , and that its thickness is ds . The intensity of the radiation just before it enters the slab is I , and after it comes out the other side, some of the radiation has been absorbed and the intensity is reduced by an amount dI . We define the opacity κ by

$$\begin{aligned} -\frac{dI}{I} &= \kappa \rho \, ds \\ \kappa &= -\frac{1}{\rho I} \frac{dI}{ds}. \end{aligned}$$

This definition makes intuitive sense: the larger a fraction of the radiation the slab absorbs (larger magnitude $(1/I)(dI/ds)$), the higher the opacity. The more material it takes to absorb a fixed amount of radiation (larger ρ), the smaller the opacity. Of course κ can depend on the frequency of the radiation in question, since some materials are very good at absorbing some frequencies and not good at absorbing others. We can imagine repeating this experiment with radiation beams at different frequencies and measuring the absorption for each one, and thus figuring out κ_ν , the opacity as a function of frequency ν .

If the radiation beam moves through a non-infinitesimal slab of uniform gas, we can calculate how much of it will be absorbed in terms of κ . Suppose the beam shining on the surface has intensity I_0 , and the slab has a thickness s . The intensity obeys

$$\frac{dI}{ds} = -\rho \kappa I,$$

so integrating we get

$$I = I_0 e^{-\kappa \rho s},$$

where the constant of integration has been chosen so that $I = I_0$ at $s = 0$. Thus radiation moving through a uniform absorbing medium is attenuated exponentially. This exponential attenuation is why common objects appear to have sharp edges – the light getting through them falls off exponentially fast. The quantity $\kappa \rho s$ comes up all the time, so we give it a special name and symbol:

$$\tau = \kappa \rho s$$

is defined as the optical depth of a system. Equivalently, we can write the differential equation describing the absorption of radiation as

$$\frac{dI}{d\tau} = -I,$$

which has the obvious solution $I = I_0 e^{-\tau}$.

B. Emission and the Radiative Transfer Equation

If radiation were only ever absorbed, life would be simple. However, material can emit radiation as well as absorb it. The equation we've written down only contains absorption, but we can generalize it quite easily to include emission. Suppose the slab of material also emits radiation at a certain rate. We describe its emission in terms of the emission rate j_ν . It tells us how much radiation energy the gas emits per unit time per unit volume per unit solid angle per unit frequency.

Including emission in our equation describing a beam of radiation traveling through a slab, we have

$$\frac{dI}{ds} = -\kappa\rho I + j.$$

The first term represents radiation taken out of the beam by absorption, and the second represents radiation put into the beam by emission. Equivalently, we can work in terms of optical depth:

$$\frac{dI}{d\tau} = -I + S,$$

where we define $S = j/(\kappa\rho)$ to be the source function. This equation is called the equation of radiative transfer.

Thus far all we've done is make formal definitions, so we haven't learned a whole lot. However, we can realize something important if we think about a completely uniform, opaque medium in thermal equilibrium at temperature T . We've already discussed that in thermal equilibrium the photons have to follow a particular distribution, called the Planck function:

$$B(\nu, T) = \left(\frac{2h\nu^3}{c^2} \right) \frac{1}{e^{h\nu/k_B T} - 1}.$$

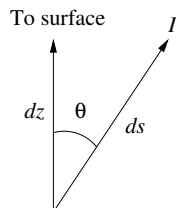
In such a medium the intensity clearly doesn't vary from point to point, so $dI/d\tau = 0$, and we have $I = S = B(\nu, T)$. This means that we know the radiation intensity and the source function for a uniform medium.

C. The Diffusion Approximation

The good news regarding stars is that, while the interior of a star isn't absolutely uniform, it's pretty close to it. There is just a tiny anisotropy coming from the fact that it is hotter toward the center of the star and colder toward the surface. The fact that the difference from uniformity is tiny means that we can write down a

simple approximation to figure out how energy moves through the star, called the diffusion approximation. It is also sometimes called the Rosseland approximation, after its discoverer, the Norwegian astrophysicist Svein Rosseland.

To derive this approximation, we set up a coordinate system so that the z direction is toward the surface of the star. For a ray at an angle θ relative to the vertical, the distance ds along the ray is related to the vertical distance dz by $ds = dz / \cos \theta$.



Therefore the transfer equation for this particular ray reads

$$\frac{dI(z, \theta)}{ds} = \cos \theta \frac{dI(z, \theta)}{dz} = \kappa \rho [S(T) - I(z, \theta)],$$

where we have written out the dependences of I and S explicitly to remind ourselves of them: the intensity depends on depth z and on angle θ , while the source function depends only on temperature T . We can rewrite this as

$$I(z, \theta) = S(T) - \frac{\cos \theta}{\kappa \rho} \frac{dI(z, \theta)}{dz}.$$

Thus far everything we have done is exact, but now we make the Rosseland approximation. In a nearly uniform medium like the center of a star, I is nearly constant, so the term $\cos \theta / (\kappa \rho) (dI(z, \theta) / dz)$ is much smaller than the term S . Thus we can set the intensity I equal to S plus a small perturbation. Moreover, since we are dealing with material that is a blackbody, S is equal to the Planck function. Thus, we write

$$I(z, \theta) = B(T) + \epsilon I^{(1)}(z, \theta),$$

where ϵ is a number much smaller than 1. To figure out what the small perturbation should be, we can substitute this approximation back into the original equation for I :

$$\begin{aligned} I(z, \theta) &= S(T) - \frac{\cos \theta}{\kappa \rho} \frac{dI(z, \theta)}{dz} \\ B(T) + \epsilon I^{(1)}(z, \theta) &= B(T) - \frac{\cos \theta}{\kappa \rho} \frac{d}{dz} [B(T) + \epsilon I^{(1)}(z, \theta)] \\ \epsilon I^{(1)}(z, \theta) &= -\frac{\cos \theta}{\kappa \rho} \frac{d}{dz} [B(T) + \epsilon I^{(1)}(z, \theta)] \\ &\approx -\frac{\cos \theta}{\kappa \rho} \frac{dB(T)}{dz}. \end{aligned}$$

In the last step, we dropped the term proportional to ϵ on the right-hand side, on the ground that $B(T) \gg \epsilon I^{(1)}(z, \theta)$. Thus we arrive at our approximate form for the intensity:

$$I(z, \theta) \approx B(T) - \frac{\cos \theta}{\kappa \rho} \frac{dB(T)}{dz}.$$

From this approximate intensity, we can now compute the radiation flux:

$$\begin{aligned} H_\nu &= \int I(z, \theta) \cos \theta d\Omega \\ &= \int \left[B(T) - \frac{\cos \theta}{\kappa \rho} \frac{dB(T)}{dz} \right] \cos \theta d\Omega \\ &= - \int \frac{\cos \theta}{\kappa \rho} \frac{dB(T)}{dz} \cos \theta d\Omega \\ &= - \frac{2\pi}{\kappa \rho} \frac{dB(T)}{dz} \int_0^\pi \cos^2 \theta \sin \theta d\theta \\ &= - \frac{2\pi}{\kappa \rho} \frac{dB(T)}{dz} \int_{-1}^1 \cos^2 \theta d(\cos \theta) \\ &= - \frac{4\pi}{3\kappa \rho} \frac{dB(T)}{dz} \\ &= - \frac{4\pi}{3\kappa \rho} \frac{\partial B(T)}{\partial T} \frac{\partial T}{\partial z} \end{aligned}$$

This is at a particular frequency. To get the total flux over all frequencies, we just integrate:

$$H = - \frac{4\pi}{3\rho} \frac{\partial T}{\partial z} \int_0^\infty \frac{1}{\kappa} \frac{\partial B(T)}{\partial T} d\nu$$

We have written all the terms that do not depend on frequency outside the integral, and left inside it only those terms that do depend on frequency. Finally, we define the Rosseland mean opacity by

$$\frac{1}{\kappa_R} \equiv \frac{\int_0^\infty \kappa^{-1} \frac{\partial B(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B(T)}{\partial T} d\nu} = \frac{\pi \int_0^\infty \kappa^{-1} \frac{\partial B(T)}{\partial T} d\nu}{4\sigma T^3},$$

where in the last step we plugged in $dB(T)/dT$ and evaluated the integral. With this definition, we can rewrite the flux as

$$H = - \frac{16\sigma T^3}{3\kappa_R \rho} \frac{\partial T}{\partial z}.$$

This is the flux per unit area. If we want to know the total energy passing through a given shell inside the star, we just multiply by the area:

$$F = -4\pi r^2 \frac{16\sigma T^3}{3\kappa_R \rho} \frac{dT}{dr}.$$

This is an extremely powerful result. It gives us the flux of energy through a given shell in the star in terms of the gradient in the temperature and the mean opacity.

We can also invert the result and write the temperature gradient in terms of the flux:

$$\begin{aligned}\frac{dT}{dr} &= -\frac{3}{16\sigma} \frac{\kappa_R \rho}{T^3} \frac{F}{4\pi r^2} \\ \frac{dT}{dm} &= -\frac{3}{16\sigma} \frac{\kappa_R}{T^3} \frac{F}{(4\pi r^2)^2}.\end{aligned}$$

This is particularly useful in the region of a star where there is no nuclear energy generation, because in such a region we have shown that the flux is constant. Thus this equation lets us figure out the temperature versus radius in a star given the flux coming from further down, and the opacity.

D. Opacity Sources in Stars

This brings us to the final topic for this class: the opacity in stars. We need to know how opaque stellar material is. There are four main types of opacity we have to worry about:

- Electron scattering: photons can scatter off free electrons with the photon energy remaining constant, a process known as Thomson scattering. The Thomson scattering opacity can be computed from quantum mechanics, and is simply a constant opacity per free electron. This constancy breaks down if the mean photon energy approaches the electron rest energy of 511 keV, but this is generally not the case in stars.
- Free-free absorption: a free electron in the vicinity of an ion can absorb a photon and go into a higher energy unbound state. The presence of the ion is critical to allowing absorption, because the potential between the electron and the ion serves as a repository for the excess energy.
- Bound-free absorption: this is otherwise known as ionization. When there are neutral atoms present, they can absorb photons whose energies are sufficient to ionize their electrons.
- Bound-bound absorption: this is like ionization, except that the transition is between one bound state and another bound state that is at a higher excitation. The $H\alpha$ and calcium K transitions we discussed a few weeks ago in the solar atmosphere are examples of this.

Which of these sources of opacity dominates depends on the local temperature and density, and changes from one part of a star to another. In the deep interior we have already seen that the gas is almost entirely ionized due to the high temperatures there, and as a result bound-free and bound-bound absorption contribute very little – there are simply too few bound electrons around. In stellar atmospheres, on the other hand, bound-free and bound-bound absorptions dominate, because there are comparatively few free electrons. For our purposes we will

mostly be concerned with stellar interiors, where electron scattering and free-free are most important.

Electron scattering is fairly easy to calculate, since it just involves the interaction of an electromagnetic wave with a single charged particle. In fact, the calculation can be done classically as long as the photon energy is much smaller than the electron rest mass. We will not do the derivation in class, and will simply quote the result. The cross-section for a single electron, called the Thomson cross-section, is

$$\sigma_T = \frac{8\pi}{3} \left(\frac{\alpha \hbar}{m_e c} \right)^2 = 6.65 \times 10^{-25} \text{ cm}^2,$$

where $\alpha \approx 1/137$ is the fine structure constant. This is independent of frequency, so the Rosseland mean opacity is simply the opacity at any frequency (except for frequencies where the photon energy approaches 511 keV.) To figure out the corresponding opacity, which is the cross-section per unit mass, we simply have to multiply this by the number of free electrons per gram of material. For pure hydrogen, there are $1/m_H$ hydrogen atoms per gram, and thus $1/m_H$ free electrons is the material is fully ionized. Thus for pure hydrogen we have

$$\kappa_{\text{es},0} = \frac{\sigma_T}{m_H} = 0.40 \text{ cm}^2 \text{ g}^{-1}.$$

If the material is not pure hydrogen we get a very similar formula, and we just have to plug in the appropriate number of hydrogen masses per free electron, μ_e . Thus we have

$$\kappa_{\text{es}} = \frac{\sigma_T}{\mu_e m_H} = \frac{\kappa_{\text{es},0}}{\mu_e} = \kappa_{\text{es},0} \left(\frac{1+X}{2} \right).$$

The other opacity source to worry about in the interior of a star is free-free absorption. This process is vastly more complicated to compute, since one must consider interactions of free electrons with many types of nuclei and with photons of many frequencies, over a wide range in temperatures and densities. The calculation these days is generally done by computer. However, we have some general expectations based on simple principles.

The idea of free-free absorption is that the presence of ions enhances the opacity of the material above and beyond what would be expected with just free electrons, because the potential energy associated with the electron-ion interaction provides a repository into which to deposit energy absorbed from photons. Since the effect depends on ion-electron interactions, a higher density should increase the opacity, since it means electrons and ions are more closely packed and thus interact more. For this reason, κ_{ff} should increase with density.

At very high temperatures, on the other hand, free-free opacity should become unimportant. This is because the typical photon energy is much higher than the electron-ion potential, so having the potential energy of the electron-ion interaction available doesn't help much. Thus κ_{ff} should decline with increasing temperature, until it becomes small compared to electron-scattering opacity.

When making stellar models on a computer, one can directly use the tables of κ_{ff} as a function of ρ and T that a computer spits out. However, we can make simple analytic models and get most of the general results right using an analytic fit to the numerical data. The free-free opacity, first derived by Hendrik Kramers, is well approximated by a power law in density and temperature:

$$\kappa_{\text{ff}} \approx \frac{\kappa_{\text{ff},0}}{\mu_e} \left\langle \frac{\mathcal{Z}^2}{\mathcal{A}} \right\rangle \rho [\text{g cm}^{-3}] T [\text{K}]^{-7/2} \approx \kappa_{\text{ff},0} \left(\frac{1+X}{2} \right) \left\langle \frac{\mathcal{Z}^2}{\mathcal{A}} \right\rangle \rho [\text{g cm}^{-3}] T [\text{K}]^{-7/2},$$

where $\kappa_{\text{ff},0} = 7.5 \times 10^{22} \text{ cm}^2 \text{ g}^{-1}$, and the square brackets after ρ and T indicate the units in which they are to be measured when plugging into this formula. As expected, the opacity increases with density and decreases with temperature. The factor $\langle \mathcal{Z}^2/\mathcal{A} \rangle$ appears because the free-free opacity is affected by the population of ions available for interactions. Interactions are stronger with more charged ions, hence the \mathcal{Z}^2 in the numerator. They are weaker with more massive ions, hence the \mathcal{A} in the denominator.

An opacity of this form, given as $\kappa = \text{const} \cdot \rho^a T^b$ for constant powers a and b , is known as a Kramers law opacity.