Astronomy 112: The Physics of Stars

Class 14 Notes: The Main Sequence

In the last class we drew a diagram that summarized the basic evolutionary path of stars, as seen from their centers. In this class we will focus on one the evolutionary phase where stars spend most of their lives: the main sequence. Our goal is to demonstrate that we can understand the main sequence qualitatively in terms of our simple stellar evolution model, and then to examine some detailed numerical results.

I. Homology and Scalings on the Main Sequence

A. The Non-Dimensional Structure Equations

We saw in the last class that stars are born at low ρ_c and T_c, and they evolve along tracks of constant mass to higher central density and pressure. This evolution takes place on a KH timescale, and ends when the stars' cores intersect the hydrogen burning line. At that point the stars ignite hydrogen and burn for a time $t_{\rm nuc} \gg t_{\rm KH}$. The hydrogen-burning nuclear timescale is the longest time for any evolutionary stage, because

$$t_{\rm nuc} = \frac{\epsilon M c^2}{L},$$

and the hydrogen burning stage is the one with the largest value of ϵ and the smallest value of L.

When we look at a population of stars that are at many different ages, and thus at many random points in their lives, we expect the number of stars we see in a given population to be proportional to the fraction of its life that a star spends as a member of that population. In other words, if one evolutionary phase lasts 1 million times longer than a second phase, we would expect to see roughly 1 million times as many stars in the first evolutionary phase than we do in the second. Since the main sequence is the most heavily populated part of the HR diagram and the hydrogen nuclear burning phase is the longest evolutionary phase, it seems natural to assume that main sequence stars are burning hydrogen.

To check this hypothesis, we need to investigate whether stars that have reached the hydrogen burning line in the $(\log T, \log \rho)$ plane have the right properties to be main sequence stars. In particular, do they have the right mass luminosity relation; and do they have the right relationship between luminosity and surface temperature?

To answer this question, we'll write down our stellar structure equations, in a simplified form to make life easy. We will neglect convection and radiation pressure, use a constant opacity, and use a powerlaw approximation for the rate of nuclear

burning. The goal is to show that we can roughly reproduce what is observed, although not get every detail right. With these assumptions, the complete set of equations is

$$\begin{aligned}
\frac{dP}{dm} &= -\frac{Gm}{4\pi r^4} \\
\frac{dr}{dm} &= \frac{1}{4\pi r^2 \rho} \\
\frac{dT}{dm} &= -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2} \\
\frac{dF}{dm} &= q_0 \rho T^\nu \\
P &= \frac{\mathcal{R}}{\mu}\rho T
\end{aligned}$$

The unknowns are $r(m)$, $P(m)$, $T(m)$, $F(m)$, and $\rho(m)$, and there are 5 equations, so the system is fully specified.

We don't have to solve the equations exactly to get out the basic behavior. Instead, we can figure out many scalings with some simple dimensional arguments. To do this, we will deploy the same technique of non-dimensionalizing the equations that we used so effectively with polytropes. We begin by defining a dimensionless mass variable

$$x = \frac{m}{M},$$

and then defining dimensionless versions of all the other variables:

$$\begin{aligned}
r &= f_1(x) R_* \\
P &= f_2(x) P_* \\
\rho &= f_3(x) \rho_* \\
T &= f_4(x) T_* \\
F &= f_5(x) F_*,
\end{aligned}$$

where M is the total mass of the star and R_*, P_*, ρ_*, T_*, and F_* are values of the radius, pressure, density, temperature and heat flow that we have not yet specified.

Thus far all we have done is define a new set of variables. We will now substitute this new set of variables into the equation of hydrostatic balance:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \qquad \longrightarrow \qquad \frac{P_*}{M}\frac{df_2}{dx} = -\frac{GMx}{4\pi R_*^4 f_1^4}.$$

We now exercise our freedom to define P_*. We define it by

$$P_* = \frac{GM^2}{R_*^4},$$

and with this choice the equation of hydrostatic balance reduces to

$$\frac{df_2}{dx} = -\frac{x}{4\pi f_1^4}.$$

This is the non-dimensional version of the equation.

We can non-dimensionalize the other equations in a similar fashion, in each case exercising our freedom to choose one of the starred quantities. For the dr/dm equation, we choose

$$\rho_* = \frac{M}{R_*^3},$$

which gives us the non-dimensional equation

$$\frac{df_1}{dx} = \frac{1}{4\pi f_1^2 f_3}.$$

For the other equations, the definitions and non-dimensionalized versions are

$$T_* = \frac{\mu P_*}{\mathcal{R}\rho_*} \qquad f_2 = f_3 f_4$$

$$F_* = \frac{ac}{\kappa}\frac{T_*^4 R_*^4}{M} \qquad \frac{df_4}{dx} = -\frac{3f_5}{4f_4^3(4\pi f_1^2)^2}$$

$$F_* = q_0 \rho_* T_*^\nu M \qquad \frac{df_5}{dx} = f_3 f_4^\nu.$$

You might be suspicious that we defined F_* twice, which we can't do. The trick is that we have yet not chosen R_*. Thus we can use our last choice to define R_* in such a way as to make the second equation here true. Once we do so, have have defined all the starred quantities, and non-dimensionalized all the equations.

What is the point of this? The trick is that the non-dimensional equations for $f_1 - f_5$ now depend only on dimensionless numbers, and not on the stellar mass. Any dependence of the solution on mass *must* enter only through the starred quantities. Another way of putting it is that these equations have the property that they are homologous – one can solve for $f_1 - f_5$, and then scale that solution to an arbitrary mass by picking a different value of M. In a sense, these equations say that all stars (for which our four assumptions above are valid) have the same structure.

Of course the only reason we were able to obtain non-dimensionalized equations of this form and demonstrate homology is due to the simplifying assumptions we made – neglect of radiation pressure, neglect of convection, adopting a constant κ, and using a powerlaw form for the nuclear energy generation rate. These complications are the basic reasons that stars do not actually all have the same structure independent of mass. Nonetheless, the first and last of these assumptions are reasonably good for low mass stars (though not for massive stars). The assumption of constant κ isn't strictly necessary, as you will demonstrate on your homework. The most questionable assumption is our neglect of convection.

B. Mass Scalings

With that aside out of the way, we can proceed to use the homologous equations to deduce the dependence of all quantities on mass. Combining the equations for ρ_*, P_* and T_* gives

$$T_* = \frac{\mu}{\mathcal{R}} \frac{GM}{R_*}.$$

Notice that we have already proven essentially this result using the virial theorem.

Inserting T_* into the equation for F_* gives

$$F_* = \frac{ac}{\kappa} \frac{R_*^4}{M} \left(\frac{\mu}{\mathcal{R}} \frac{GM}{R_*} \right)^4 = \frac{ac}{\kappa} \left(\frac{\mu G}{\mathcal{R}} \right)^4 M^3$$

Since this relation applies at any value of x, it must apply at $x = 1$, i.e. at the surface of the star. Since at the stellar surface $L = F = F_* f_5(1)$, it immediately follows that

$$L \propto \frac{ac}{\kappa} \left(\frac{\mu G}{\mathcal{R}} \right)^4 M^3.$$

Thus the luminosity varies as M^3. Notice that this is independent of any of the other starred quantities – we have derived the dependence of L on the mass alone. Also notice that this result is basically the same as we get from Eddington's model with $\beta = 1$ (i.e. our assumption of no radiation pressure) – which makes sense, since Eddington's model is a polytrope, and therefore homologous, and also has constant κ. Thus we couldn't possibly find anything else.

We can now push further and deduce the mass scalings of other quantities as well. We have

$$F_* = q_0 \rho_* T_*^\nu M = \frac{ac}{\kappa} \left(\frac{\mu G}{\mathcal{R}} \right)^4 M^3 \qquad \Longrightarrow \qquad \rho_* = \frac{ac}{q_0 \kappa} \left(\frac{\mu G}{\mathcal{R}} \right)^4 \frac{M^2}{T_*^\nu}.$$

Substituting for ρ_* and T_* gives

$$\frac{M}{R_*^3} = \frac{ac}{q_0 \kappa} \left(\frac{\mu G}{\mathcal{R}} \right)^4 M^2 \left(\frac{\mu P_*}{\mathcal{R} \rho_*} \right)^{-\nu}$$

Finally, substituting for P_* and ρ_* again gives

$$\begin{aligned}
\frac{M}{R_*^3} &= \frac{ac}{q_0 \kappa} \left(\frac{\mu G}{\mathcal{R}} \right)^4 M^2 \left(\frac{\mu}{\mathcal{R}} \right)^{-\nu} \left(\frac{GM^2}{R_*^4} \frac{R_*^3}{M} \right)^{-\nu} \\
\frac{M}{R_*^3} &= \frac{ac}{q_0 \kappa} \left(\frac{\mu G}{\mathcal{R}} \right)^{4-\nu} M^{2-\nu} R_*^\nu \\
R_* &= \left[\frac{q_0 \kappa}{ac} \left(\frac{\mathcal{R}}{\mu G} \right)^{4-\nu} \right]^{1/(\nu+3)} M^{(\nu-1)/(\nu+3)}
\end{aligned}$$

Thus we expect the stellar radius to scale with mass in a way that depends on how the nuclear reactions scale with temperature. If we have a star that burns

hydrogen mainly via the *pp* chain, then $\nu \approx 4$, and we obtain $R \propto M^{3/7}$. For a more massive star that burns mainly via the CNO cycle, we have $\nu \approx 20$, and we instead obtain $R \propto M^{19/23}$, a nearly linear relationship. Thus we expect the radius to increase with mass as $M^{3/7}$ at small masses, increasing in steepness to a nearly linear relationship at larger masses.

For the density, we have

$$\rho_* = \frac{M}{R_*^3} = \left[\frac{q_0 \kappa}{ac}\left(\frac{\mathcal{R}}{\mu G}\right)^{4-\nu}\right]^{-3/(\nu+3)} M^{2(3-\nu)/(3+\nu)}.$$

For *pp* chain stars, this gives $\rho_* \propto M^{-2/7}$, and for CNO cycle stars it gives $\rho_* \propto M^{-34/23}$, which is nearly -1.5. Thus the density always decreases with increasing stellar mass, but does so fairly slowly for *pp* chain stars (-0.29 power) and quite rapidly for CNO cycle stars (-1.5 power). This is an important and often under-appreciated point in stellar structure: more massive stars are actually much less dense than less massive ones. Very massive stars are quite puffy and diffuse.

C. The Observed Main Sequence

Finally, we can get out the scaling that we really care about: luminosity versus temperature. This is what will determine the shape of the observed main sequence, and we had better make sure that what we get out of the theoretical model agrees reasonably well with what we actually observe. If not, the hypothesis that the main sequence is made up of stars whose cores are stalled on the hydrogen burning line will not be valid.

The effective temperature is related to the radius and luminosity by

$$\frac{L}{4\pi R^2 \sigma} = T_{\rm eff}^4.$$

However we have just shown that

$$L \propto M^3 \qquad \text{and} \qquad R \propto M^{(\nu-1)/(\nu+3)}.$$

Inverting the first relation and substituting it into the second, we have

$$M \propto L^{1/3} \qquad \Longrightarrow \qquad R \propto \left(L^{1/3}\right)^{(\nu-1)/(\nu+3)} \propto L^{(\nu-1)/[3(\nu+3)]}.$$

Now plugging this into the relationship between L and $T_{\rm eff}$, we give

$$\begin{aligned}
\frac{L}{\left[L^{(\nu-1)/[3(\nu+3)]}\right]^2} &\propto T_{\rm eff}^4 \\
L^{1-2(\nu-1)/[3(\nu+3)]} &\propto T_{\rm eff}^4 \\
\left[1 - \frac{2(\nu-1)}{3(\nu+3)}\right] \log L &= 4 \log T_{\rm eff} + \text{constant} \\
\log L &= 4\left[1 - \frac{2(\nu-1)}{3(\nu+3)}\right]^{-1} \log T_{\rm eff} + \text{constant}.
\end{aligned}$$

We have therefore derived an equation for the slope that the main sequence should have in the HR diagram, which shows $\log L$ vs. $\log T_{\rm eff}$.

Plugging in $\nu = 4$ for *pp* chain stars and $\nu = 20$ for CNO cycle stars, we obtain

$$\begin{aligned} \log L &= 5.6 \log T_{\rm eff} + \text{constant} \qquad (pp \text{ chain}) \\ \log L &= 8.9 \log T_{\rm eff} + \text{constant} \qquad (\text{CNO cycle}) \end{aligned}$$

The values compare reasonably well with the observed slopes of the lower and upper main sequence on the HR diagram.

We can also explain other features of observed HR diagrams with this simple model. As we noted when we discussed star clusters of different ages, more massive, luminous stars leave the main sequence before lower mass stars. In the picture we have now developed, leaving the main sequence corresponds to moving past the hydrogen ignition point in the $(\log T, \log \rho)$ plane. The time spent at that point is given roughly by the nuclear timescale,

$$t_{\rm nuc} = \frac{\epsilon M c^2}{L} \propto M^{-2}$$

This means that the nuclear timescale should decrease with stellar mass as M^{-2}, so more massive stars have shorter nuclear timescales and leave the main sequence first. This is exactly what we observe.

We can also use this analysis to estimate the behavior of the upper and lower ends of the main sequence, by scaling from the Sun. The temperature in the star scales as

$$T_* \propto \frac{M}{R_*} \propto \frac{M}{M^{(\nu-1)/(\nu+3)}} \propto M^{4/(\nu+3)}.$$

For *pp* chain stars, this means that $T_* \propto M^{4/7}$. This applies throughout the star, including in the center.

The Sun has a central temperature of $T_{c,\odot} \simeq 1.5 \times 10^7$ K, so we expect that

$$\frac{T_c}{T_{c,\odot}} = \left(\frac{M}{M_\odot}\right)^{4/7}.$$

Thus lower mass stars than the Sun have lower central temperatures. However, the temperature cannot decline indefinitely without interfering with the star's ability to generate energy. If $T_c \lesssim 4 \times 10^6$ K, then the star will not be able to burn hydrogen, and it cannot ignite. The mass at which this limit is reached is

$$M = M_\odot \left(\frac{4 \times 10^6 \text{ K}}{1.5 \times 10^7 \text{ K}}\right)^{7/4} = 0.1 M_\odot.$$

The corresponding luminosity for a star at this limit is

$$L = L_\odot \left(\frac{M}{M_\odot}\right)^3 = 10^{-3} L_\odot.$$

Thus the dimmest stars in existence should have a luminosity about 1/1000th that of the Sun.

Of course this argument is a bit of a cheat: the way we got the homology argument in the first place is by assuming a particular scaling for the nuclear reaction rate with temperature ν, and, if the temperature gets too low, ν will change. Thus the limit is a bit more complicated. Nonetheless, this argument gives a reasonable estimate for the minimum mass of an object that can burn hydrogen. Lower mass objects never ignite hydrogen, and instead end up being supported by degeneracy pressure. This objects are called brown dwarfs, and, even though they form like stars, they end up having structures more like that of the planet Jupiter.

In the opposite direction, we can use a similar scaling argument to deduce when the mass of a star becomes large enough for it to approach the Eddington limit. The star's luminosity scales like $L \propto M^3$, and the Eddington luminosity is

$$L_{\rm Edd} = \frac{4\pi c G M}{\kappa} \propto M.$$

Thus

$$\frac{L}{L_{\rm Edd}} \propto M^2.$$

Again scaling from the Sun, we find

$$\frac{L/L_{\rm Edd}}{L_\odot/L_{\rm Edd,\odot}} = \left(\frac{M}{M_\odot}\right)^2.$$

Thus the mass at which the luminosity reaches the Eddington luminosity is

$$M = M_\odot \left(\frac{1}{L_\odot/L_{\rm Edd,\odot}}\right)^{1/2} = M_\odot \left(\frac{4\pi c G M_\odot}{\kappa L_\odot}\right)^{1/2} = 114 M_\odot.$$

The corresponding luminosity is

$$L = L_\odot \left(\frac{M}{M_\odot}\right)^3 = 1.5 \times 10^6 L_\odot.$$

Thus the most luminous stars in existence should be more than a million times as bright as the Sun.

Of course this calculation too is a bit of a cheat – the equations we used to derive the scalings on which this relationship is based assume negligible radiation pressure, which is obviously not the case in a star that is approaching the Eddington limit. Nonetheless, this again gives us a rough idea of where we should start crossing over into stars that are supported mostly by radiation. Since we have seen that these stars tend to have stability problems, we do not expect to find a lot of stars of this mass, and thus we expect this to represent a rough upper limit to the main sequence.

D. Convective Stars

The calculation we just performed is for stars where energy is transported radiatively. However, we said last week that there are parts of stars where instead the energy is transported convectively. We can repeat this homology analysis for the case of convective stars, and derive the main sequence mass-luminosity and temperature-luminosity relation we expect in that case to compare to our radiative results. Since real stars are usually convective over part but not all of their interiors, the real behavior should lie in between these extremes.

We showed that convection should bring the temperature gradient within a star extremely close to the adiabatic temperature gradient, so we can to very good approximation say that the gas is adiabatic, and the star is therefore a polytrope with $\gamma_P = \gamma_a = \gamma$. This implies that

$$P = K_a \rho^\gamma$$

and

$$T = \frac{\mu}{\mathcal{R}} \frac{P}{\rho} = \frac{\mu}{\mathcal{R}} K_a \rho^{\gamma-1} = \frac{\mu}{\mathcal{R}} K_a \left(\frac{P}{K_a}\right)^{(\gamma-1)/\gamma} = \frac{\mu}{\mathcal{R}} K_a^{1/\gamma} P^{(\gamma-1)/\gamma}$$

For an ideal non-relativistic gas, $\gamma = 5/3$. These equations replace the ideal gas law and the radiation diffusion equation in our set of stellar structure equations. The other equations are unchanged:

$$\begin{aligned} \frac{dP}{dm} &= -\frac{Gm}{4\pi r^4} \\ \frac{dr}{dm} &= \frac{1}{4\pi r^2 \rho} \\ \frac{dF}{dm} &= q_0 \rho T^\nu \end{aligned}$$

The procedure for non-dimensionalizing these equations is essentially the same as in the radiative case. We make the same changes of variables and in all equations except the temperature one, and get the set of non-dimensional equations

$$\begin{aligned} P_* &= \frac{GM^2}{R_*^4} & \frac{df_2}{dx} &= -\frac{x}{4\pi f_1^4} \\ \rho_* &= \frac{M}{R_*^3} & \frac{df_1}{dx} &= \frac{1}{4\pi f_1^2 f_3} \\ P_* &= K_a \rho_*^\gamma & f_2 &= f_3^\gamma \\ T_* &= \frac{\mu}{\mathcal{R}} K_a^{1/\gamma} P_*^{(\gamma-1)/\gamma} & f_4 &= f_2^{(\gamma-1)/\gamma} \\ F_* &= q_0 \rho_* T_*^\nu M & \frac{df_5}{dx} &= f_3 f_4^\nu . \end{aligned}$$

To get the mass-luminosity and effective-temperature luminosity relations in this case, we have the same basic problem as in the radiative case: given this set

of 5 algebraic equations in 5 variables, we must manipulate them to get M by itself on the right-hand side. We're only interested in the proportionalities, so for convenience we can take the logarithm of everything and forget about the constants:

$$\begin{aligned}
\log P_* &= 2\log M - 4\log R_* + \text{constant} \\
\log \rho_* &= \log M - 3\log R_* \\
\log P_* &= \gamma \log \rho_* + \text{constant} \\
\log T_* &= \left(\frac{\gamma-1}{\gamma}\right)\log P_* + \text{constant} \\
\log F_* &= \log \rho_* + \nu \log T_* + \log M + \text{constant}.
\end{aligned}$$

We now have the same algebra problem as in the radiative case, which is to rearrange this set of 5 equations so that everything is given in terms of M. This is not hard, since in this logarithmic form the equations are linear, and this just represents a set of 5 linear equations in 5 unknowns. To solve, we begin by equating the two expressions for $\log P_*$ and substituting for ρ_*:

$$\begin{aligned}
2\log M - 4\log R_* &= \gamma \log \rho_* + \text{constant} \\
2\log M - 4\log R_* &= \gamma(\log M - 3\log R_*) + \text{constant} \\
\log R_* &= \left(\frac{\gamma-2}{3\gamma-4}\right)\log M + \text{constant}.
\end{aligned}$$

Thus we now have $\log R_*$ in terms of $\log M$ alone. We now substitute this into the equations for $\log P_*$, $\log \rho_*$, and $\log T_*$ to solve for them in terms of M:

$$\begin{aligned}
\log P_* &= \left(\frac{2\gamma}{3\gamma-4}\right)\log M + \text{constant} \\
\log \rho_* &= \left(\frac{2}{3\gamma-4}\right)\log M + \text{constant} \\
\log T_* &= \left(\frac{2(\gamma-1)}{3\gamma-4}\right)\log M + \text{constant}.
\end{aligned}$$

We then substitute this into the equation for $\log F_* = \log L + \text{constant}$:

$$\log L = \left(\frac{\gamma(2\nu+3) - 2(\nu+1)}{3\gamma-4}\right)\log M + \text{constant}$$

This gives the mass luminosity relation. For $\gamma = 5/3$ and $\nu = 4$, the constant is $25/3 = 8.33$, to the luminosity varies very steeply with mass.

The final step is to turn this into a luminosity temperature relation, using

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4 \qquad \Longrightarrow \qquad \log L = 2\log R + 4\log T_{\text{eff}} + \text{constant}.$$

We therefore need $\log R$ in terms of $\log L$. To get it, we use our expressions for both $\log R_*$ and $\log L_*$ in terms of $\log M$, which give:

$$\log R_* = \left(\frac{\gamma - 2}{\gamma(2\nu + 3) - 2(\nu + 1)} \right) \log L + \text{constant}.$$

Plugging this into the expression for $\log R$ in the luminosity-effective temperature relationship and solving finally gives

$$\log L = \left(\frac{4(\gamma(2\nu + 3) - 2(\nu + 1))}{2\nu(\gamma - 1) + \gamma + 2} \right) \log T_{\text{eff}} + \text{constant}.$$

This is the luminosity-effective temperature relationship for a polytropic star with arbitrary γ. For a fully convective star composed of non-relativistic ideal gas, $\gamma = 5/3$, and for $p - p$ chain burning, $\nu = 4$. Plugging in these values gives a value of 3.7 for the coefficient. This is shallower than the value of 5.6 we found for radiative stars with constant κ. As you will show on your homework, the value for radiative stars where κ is the free-free opacity differs slightly from these.

II. Numerical Results on the ZAMS

We have now pushed as far as we are going to analytically, and the time has come to bring out the computers. We have written down all the necessary equations, and they can be solved by modern computers quite easily. We will not discuss the necessary algorithms – that is a main topic of a graduate stellar structure class. Instead, we will simply review the important results. For today we will focus on stars that have not yet processed a significant amount of hydrogen into helium. Stars of this sort are said to be on the zero age main sequence, or ZAMS for short. We will talk about evolution of stars before and after the ZAMS next week.

A. Mass-Luminosity-Effective Temperature Relations

The most basic output of the numerical codes is a prediction for the luminosity and effective temperature of a star of a given mass and composition. The figures show the results of one particular set of numerical stellar models that is freely downloadable on the web. These are called the Geneva models, since the research group that produced them is centered at Geneva Observatory.

[Slides 1 and 2 – mass-luminosity and luminosity-effective temperature relations]

The basic behavior is essentially as we predicted from our simple models (the no convection model does better). The luminosity scales as mass to roughly the third power at low masses – slightly steeper due both to the effects of convection and the varying opacity. At higher masses the dependence flattens out, approaching $L \propto M$ at the very highest masses, The most massive stars have luminosities of a bit more than $10^6\ L_\odot$, while the lowest mass ones are below $10^{-2}\ L_\odot$ – these tracks only go down to $0.4\ M_\odot$, so they don't probe what are really the absolute smallest stars.

Similarly, the plot of $\log L$ vs. $\log T_{\rm eff}$ has a slope of $\sim 5-6$ for intermediate mass stars, with values of the effective temperature ranging from a few thousand Kelvin to several tens of thousands.

B. Interior Structure and Convection

Another basic output of the numerical models is a prediction for the internal structure of stars, meaning the run of density, mass, temperature, pressure, etc. versus radius. The plot shows an example for the Sun. We see that stars are very centrally concentrated – the density and pressure fall to $\sim 1\%$ of their central value by a radius of 50% of the total stellar radius. Thus one may reasonably think of the Sun and other stars as consisting of a compact, dense core, surrounded by a fluffy, diffuse envelope.

[Slide 3 – Solar properties versus radius]

Another useful plot is one that compares the structures of stars of different masses.

[Slide 4 – convection and structure versus mass]

This type of plot is a little complicated, because it packs in a lot of information, but it is very useful. Here's how to read it: the x axis is the mass of star we're examining. The y axis indicates position within the star, using Lagrangian coordinates. Thus the core of the star is at the bottom, $m/M = 0$, and the surface is at the top, $m/M = 1$.

Within this coordinate system, one can draw lines like the ones in the figure labelled with $0.5R$. This curve shows, for a star of mass M on the x axis, what fraction of the mass (m/M) is within a radius that is 50% of the total stellar radius. As the plot shows, this mass fraction (m/M) is usually significantly larger than 0.5, and is often well above 0.8. This is telling us that stars are quite centrally concentrated: the inner half of the radius contains the great majority of the mass.

Similarly the curves labelled with $0.5L$ and $0.9L$ indicate where in the star the luminosity is being generated. The line $0.9L$ indicates the mass that is responsible for generating 90% of the total power. This is quite close to the center, $m/M \sim 0.1-0.2$, particularly for massive stars. This is because nuclear burning is very temperature-sensitive, particularly for the CNO cycle, so most burning happens near the center where the temperature is highest.

Finally, the regions with the circles shows where convection occurs in the star. In low mass stars, convection occurs near the surface, where it is driven by a combination of partial ionziation (which produces γ_a near 1) and low temperature (which produces large opacity). In massive stars, convection occurs near the center, where the high temperature sensitivity of the CNO cycle causes the heat flow to change very rapidly with radius.

Notice that stars below about 0.3 $M_\odot$ are fully convective – and thus are well described by $n = 1.5$ polytropes. The amount of convective mass also increases

with mass in very massive stars, as rapid nuclear burning causes a larger and larger temperature gradient. In contrast, the Sun lies near a minimum in terms of the fraction of its mass that is convective. Only a few percent of the Sun's mass lies in the convection zone, although it is a considerably larger fraction of the Sun's radius.

C. Main Sequence Lifetime

Knowing where the star is convective is particularly important for a number of reasons. Perhaps the biggest one is that the size of the convective zone influences the amount of fuel available to the star, and thus the main sequence lifetime.

Within a convection zone, the churning of mass up and down tends to mix the gas, and homogenize the composition – convection acts like a giant paint mixer. In stars with convective cores, the stellar paint mixer has the important effect of dragging fresh hydrogen down into the nuclear burning core, supplying additional fuel. In contrast, in stars like the Sun that are not convective in their cores, the supply of hydrogen is limited to what was present at a given radius when the star formed – there is no effective way to bring in extra hydrogen from further up in the star.

We previously estimated the nuclear timescale as

$$t_{\rm nuc} = \frac{\epsilon M c^2}{L} = 100 \frac{M/M_\odot}{L/L_\odot} \text{ Gyr}$$

where $\epsilon = 6.6 \times 10^{-3}$ for hydrogen burning. However, now we understand a subtlety in this estimate that we did not before: only gas within the nuclear burning region, or within a convective zone that includes it, is available as fuel. Thus the mass that enters this timescale estimate should only be that fraction of the mass that is available for burning.

This has important implications for the Sun. Unmodified, our estimate would suggest that the Sun should have a 100 Gyr lifetime on the main sequence. However, we have mentioned several times that this is an overestimate, and that the true answer is closer to 10 Gyr. The reason for this is that the Sun is not convective over the vast majority of its mass, so only mass within the nuclear burning region should be counted in computing $t_{\rm nuc}$. Examining the structure plot, we see that for a 1 $M_\odot$ star, half the nuclear energy is produced within the central 10% of the mass, so the mass available for nuclear burning is $\sim 0.1\ M_\odot$, not $\sim 1\ M_\odot$.. This is why the Sun's main sequence lifetime is roughly 10 Gyr, not 100 Gyr.

We will return to the question of main sequence lifetimes, and what happens to stars after they have exhausted their hydrogen supplies, next week.