Before moving on to make more sophisticated models of stars than those discussed in the last week, we need to check that our models are stable. What this means is that we need to check not only that we have found solutions to the equations of stellar structure that are steady in time (since we dropped the time derivatives), we need to check that those solutions have the property that small perturbations, which are always present, tend to damp out and return the system back to the equilibrium we have identified. This is what is mean by stability. In contrast, instability occurs when any small deviation from an equilibrium solution tends to drive the system further and further away from it.

The classic example of an unstable system is a pencil standing on its point. If one could get the pencil to balance completely perfectly, it would be in equilibrium. However, any small perturbation that causes the pencil to tilt slightly will grow, and the pencil will fall over. We need to make sure that our solutions to the stellar structure equations aren’t like a pencil standing on end, or, if they are, to understand why and what that implies.

I. Stability of Nuclear Burning

We begin our consideration of stability in stars by examining thermal stability. That is, we recall that \( t_{\text{dyn}} \ll t_{\text{KH}} \ll t_{\text{nuc}} \), and for now we neglect instabilities that occur on the dynamical timescale. We assume that the star is in dynamical equilibrium, and we do not worry about the stability of our solution to the equation of hydrostatic balance. Instead, we worry about the stability of our solutions to the equations describing energy generation and transport.

A. Non-Degenerate Ideal Gas with Radiation

One way to approach this problem is to consider a star as a whole and apply the virial theorem. Consider a star that is supported by a combination of ideal, non-degenerate, non-relativistic gas pressure and radiation pressure, so that

\[
P = \frac{\mathcal{R}}{\mu} \rho T + \frac{1}{3} aT^4.
\]

In terms of internal energy, recall that the specific internal energies of gas and radiation are given by

\[
\begin{align*}
 u_{\text{gas}} &= \frac{3}{2} \frac{\mathcal{R}}{\mu} T = \frac{3}{2} \frac{P_{\text{gas}}}{\rho}, \\
 u_{\text{rad}} &= \frac{aT^4}{\rho} = 3 \frac{P_{\text{rad}}}{\rho}.
\end{align*}
\]

The virial theorem tells us that pressure and gravitational binding energy are related by

\[
\Omega = -3 \int_0^M \frac{P}{\rho} \, dm,
\]
so plugging in the pressure gives
\[ \Omega = -3 \int_0^M \left( \frac{2}{3} u_{\text{gas}} + \frac{1}{3} u_{\text{rad}} \right) \, dm = -(2U_{\text{gas}} + U_{\text{rad}}), \]
where \( U_{\text{gas}} \) and \( U_{\text{rad}} \) are the total internal energies of gas and radiation in the star. Thus
\[ U_{\text{gas}} = -\frac{1}{2}(\Omega + U_{\text{rad}}) \]
and the total energy is
\[ E = \Omega + U_{\text{rad}} + U_{\text{gas}} = -U_{\text{gas}}. \]

Thus the total energy of the star is \(-U_{\text{gas}}\). What does this imply about the mean temperature in the star? Recall that if \( T \) is the mass-averaged temperature in the star, the gas internal energy is
\[ U_{\text{gas}} = \frac{3}{2} M \frac{\mathcal{R}}{\mu} T. \]
Conservation of energy for a star therefore requires that
\[ L_{\text{nuc}} - L = \frac{dE}{dt} = -\frac{3}{2} M \frac{\mathcal{R}}{\mu} \frac{dT}{dt}, \]
where \( L_{\text{nuc}} \) is the total rate of nuclear energy generation in a star, and \( L \) is its total luminosity. This assumes \( M \) and \( \mu \) are constant over the time we are considering.

In thermal equilibrium \( L_{\text{nuc}} = L \) and the left-hand side vanishes. To investigate stability, consider what would happen if \( L_{\text{nuc}} \) and \( L \) were slightly different, so that \( L_{\text{nuc}} - L = \delta L \neq 0 \). In this case the mean temperature would change according to
\[ \frac{dT}{dt} = -\frac{2}{3} \frac{\mu}{\mathcal{R} M} \delta L. \]
Thus if \( \delta L > 0 \), meaning that \( L_{\text{nuc}} > L \), then \( \frac{dT}{dt} < 0 \), and the temperature decreases. Since, as we have seen, \( L_{\text{nuc}} \) is a strongly increasing function of \( T \), this means that \( L_{\text{nuc}} \) will in turn decrease, and \( \delta L \) will decrease too. Conversely, if \( \delta L < 0 \), then the temperature will increase, \( L_{\text{nuc}} \) will rise, and \( \delta L \) will increase. Thus an imbalance in one direction creates a restoring force in the opposite direction. This is the hallmark of a stable system. Stars supported by non-degenerate ideal gas plus radiation pressure are therefore thermally stable.

It is worth pausing to note that this result is actually somewhat counterintuitive, and it arises because gravity is a strange force. If \( \delta L > 0 \), this means that the star is generating more energy than it is radiating. If one considers an ordinary object that is producing more heat than it is radiating, one expects it to heat up – when one starts a fire, the fireplace gets hot because it is producing more heat than it is radiating. A star, however, does exactly the opposite. If it produces more
heat than it radiates, it actually gets colder, not hotter. This is a generic feature of systems that are held together by gravity: adding energy to a self-gravitating system makes it colder, not hotter, exactly the opposite of our experience in everyday life. The reason this happens is that gravity is an attractive force, and this causes self-gravitating systems to have a negative specific heat: adding heat makes them colder. Systems we’re used to deal with in everyday life do not have strong long-range attractive forces, and as a result they have positive specific heat. Adding heat makes them hotter.

B. Degenerate Ideal Gas with Radiation

Now let us extend this analysis to a degenerate ideal gas. For a non-relativistic degenerate gas, recall that we showed that the internal energy is related to pressure and density exactly as for a non-degenerate gas: $u_{\text{gas}} = \frac{3}{2} \frac{P_{\text{gas}}}{\rho}$, where now $P_{\text{gas}}$ is the degeneracy pressure. Consequently, our calculation of the total energy and the effects of radiation using the virial theorem is unchanged:

$$E = -U_{\text{gas}} \implies L_{\text{nuc}} - L = -\frac{dU_{\text{gas}}}{dt}.$$ 

The difference that degeneracy makes is that now $U_{\text{gas}}$ does not depend on the gas temperature, because $P_{\text{gas}}$ does not depend on temperature for a degenerate gas. Consequently, if $L_{\text{nuc}}$ and $L$ are out of balance the star can expand or contract (since $P$ and $\rho$ can change), but this does not cause the temperature to change. The temperature instead will respond only to the local rate of energy generation.

This is an unstable situation. Suppose there is a fluctuation in which $L_{\text{nuc}} > L$. The star will expand, but the temperature will not drop as a result; instead, it will rise, responding to the increase in the local rate of energy generation. As a result $L_{\text{nuc}}$ will increase rather than decrease, pushing the star further out of balance, consuming nuclear fuel even faster. This is called a thermonuclear runaway, and it leads to a phenomenon called novae that occur on the surfaces of white dwarf stars. It is also important for the evolution of red giant stars.

The runaway ends once the temperature becomes high enough that the star is no longer degenerate. Once degeneracy ends, the temperature no longer increases for $L_{\text{nuc}} > L$. Instead, it decreases, as in a non-degenerate star, and the situation is stabilized. This is called lifting the degeneracy.

One can write down the condition for instability somewhat more rigorously by considering the center of the star. Recall that polytropes obey

$$P_c = (4\pi)^{1/3} B_n GM^{2/3} \rho_c^{4/3},$$

where $P_c$ and $\rho_c$ are the central pressure and density, and $B_n$ is a number that depends on the polytropic index $n$, but only very weakly. We therefore expect a relation of this sort to hold approximately even in stars that are not perfect polytropes. This relation is only true in hydrostatic equilibrium, but we are assuming
hydrostatic equilibrium for now, since we are only concerned with instabilities on a Kelvin-Helmholtz timescale, not a dynamical timescale.

Now consider a perturbation that causes the central density to change by an amount $d\rho_c$ over a time $dt$. The corresponding change in pressure is given by

$$\frac{dP_c}{dt} = \left(4\pi\right)^{1/3} B_c G M^{2/3} \frac{4}{3} \rho_c^{1/3} \frac{d\rho_c}{dt}.$$  

Dividing this equation by the previous one gives

$$\frac{dP_c}{P_c} = \frac{4}{3} \frac{d\rho_c}{\rho_c}.$$  

The pressure and density are related by the equation of state $P(\rho, T)$. We can write a general equation of state near some particular density and pressure as

$$P = P_0 \rho^a T^b,$$

where $a$ and $b$ are numbers that depend on the type of gas. A non-degenerate gas has $a = 1$ and $b = 1$, a degenerate non-relativistic gas has $a = 5/3$ and $b = 0$, and a degenerate relativistic gas has $a = 4/3$, $b = 0$.

Again, let us consider perturbing the pressure by a small amount $dP$. The density and temperature respond according to

$$dP = P_0 \left( a \rho^{a-1} T^b d\rho + b \rho^a T^{b-1} dT \right).$$

Dividing this by the equation of state gives

$$\frac{dP}{P} = a \frac{d\rho}{\rho} + b \frac{dT}{T}.$$  

If we apply this relation at the center of the star and substitute in our result based on hydrostatic balance for $dP_c/P_c$, we get

$$\frac{4}{3} \frac{d\rho_c}{\rho_c} = a \frac{d\rho_c}{\rho_c} + b \frac{dT_c}{T_c} \quad \frac{b}{T_c} = \frac{4}{3} - a \frac{d\rho_c}{\rho_c}.$$  

Consider what this implies for various types of gas. For a non-degenerate gas, $a = 1$ and $b = 1$, the coefficients on both sides are positive, and this means that an increase in density causes an increase in temperature. This means that contraction of the star, which raises $\rho_c$, also raises the temperature. This increases the rate of nuclear burning, raising the pressure and causing the star to re-expand. Conversely, expansion of the star lowers $\rho_c$ and thus also lowers $T_c$. This reduces the rate of nuclear burning and causes the star to stop expanding.
For a degenerate gas, on the other hand, \( a \geq 4/3 \) (depending on how relativistic the gas is) and \( b \ll 1 \) (reaching 0 exactly for a perfectly degenerate gas). Thus the coefficients on the left and right sides have opposite signs. As a result, expansion of the star \( (d\rho_c < 0) \) raises the temperature \( (dT_c > 0) \), and the rate of nuclear burning increases. This pushes the star to expand even further, and leads to an unstable runaway that ends only once the gas is hot enough to drive \( a \) back below 4/3.

C. Thin Shell Instability

The two cases we have considered thus far are for entire stars. However, it sometimes occurs that nuclear burning takes place not in an entire star, but in a thin shell within it. This often happens in evolved stars that have used up their hydrogen fuel. The center of the star fills with ash supported by degeneracy pressure that the star is too cool to burn further, but on top of this ash layer there is still fuel left and burning continues. In this case the burning is generally confined to a thin shell on top of the degenerate ash core.

Consider such a burning shell of mass \( dm \), temperature \( T \), density \( \rho \), outer radius \( r_{sh} \), and inner radius \( r_0 \), which is taken to be fixed, as will be the case for a shell supported by a degenerate ash core. The thickness is \( dr \), which is much less than the radius of the star, \( R \).

The star is in hydrostatic equilibrium, so the pressure in the shell is determined by the equation of hydrostatic balance:

\[
\frac{dP}{dm} = -\frac{GM}{4\pi r^4}.
\]

Thus the pressure in the shell is given by

\[
P_{sh} = -\int_{m_{sh}}^{M} Gm \left( \frac{4\pi r^4}{4\pi r^4} \right) dm,
\]

where \( m_{sh} \) is the mass interior to the shell.

Now consider perturbing the shell by changing its pressure by an amount \( \delta P \). We would like to know the corresponding amount \( \delta r \) by which the outer radius \( r_{sh} \) of the shell changes. In hydrostatic equilibrium the star should behave homologously, meaning that the radius of every shell simply rises in proportion to the amount by which the outer edge of the perturbed shell expands. The outer edge of the perturbed shell expands by a factor \( 1 + \delta r/r_{sh} \), so a shell of gas that was at radius \( r \) moves to radius \( r(1 + \delta r/r_{sh}) \) after the perturbation.

Since hydrostatic balance still holds, the new pressure in the shell is

\[
P_{sh} + \delta P \approx -\int_{m_{sh}}^{M} \frac{Gm}{4\pi r(1 + \delta r/r_{sh})^4} dm = -\left(1 + \frac{\delta r}{r_{sh}}\right)^{-4} \int_{m_{sh}}^{M} \frac{Gm}{4\pi r^4} dm.
\]

For small perturbations, \( \delta r \ll r_{sh} \), we can use a Taylor expansion to approximate the term in parentheses, and drop higher-order terms on the grounds that they
are small:

\[ P_{\text{sh}} + \delta P = - \left( 1 - 4 \frac{\delta r}{r_{\text{sh}}} \right) \int_{m_{\text{sh}}}^{M} \frac{Gm}{4\pi r^4} \, dm' \]

\[ \delta P = 4 \frac{\delta r}{r_{\text{sh}}} \int_{m_{\text{sh}}}^{M} \frac{Gm}{4\pi r^4} \, dm \]

\[ \frac{\delta P}{P_{\text{sh}}} = -4 \frac{\delta r}{r_{\text{sh}}}. \]

This procedure of Taylor expanding in the small perturbation and dropping the higher-order terms is known as linearization, and it is one of the most powerful techniques available for analyzing differential equations.

Given this result, we can figure out how the density in the shell responds to the perturbation. The shell density is the ratio of its mass to its volume:

\[ \rho = \frac{dm}{4\pi r_{\text{sh}}^2 dr}. \]

We want to know the amount \( \delta \rho \) by which the density changes. After the perturbation, the new thickness of the shell is \( dr + \delta r \), so the new density is

\[ \rho + \delta \rho = \frac{dm}{4\pi r_{\text{sh}}^2 (dr + \delta r)} = \frac{dm}{4\pi r_{\text{sh}}^2 dr} \left( 1 + \frac{\delta r}{dr} \right)^{-1} = \rho \left( 1 + \frac{\delta r}{dr} \right)^{-1}. \]

If we now do the same trick of Taylor-expanding the \( (1 + \delta r/dr) \) term, we have

\[ \rho + \delta \rho \approx \rho \left( 1 - \frac{\delta r}{dr} \right) \quad \implies \quad \frac{\delta \rho}{\rho} = -\frac{\delta r}{dr} = -\frac{\delta r}{r_{\text{sh}} dr}. \]

Combining this result for the perturbed density with the one for the perturbed pressure, we have

\[ \frac{dP}{P_{\text{sh}}} = 4 \frac{dr}{r_{\text{sh}}} \frac{d\rho}{\rho}. \]

Before going on, we can pause to understand the physical significance of this equation. Notice that, for a thin shell, \( dr/r_{\text{sh}} \) is a very small number. This means that, for a given fractional change in density \( \delta \rho/\rho \), the fractional change in pressure \( dP/P \), is much smaller. This makes intuitive sense. Suppose we have a thin shell and we double its thickness. The shell’s volume goes up by a factor of two, so its density drops by a factor of two. On the other hand, the pressure is dictated by the weight of the overlying material, which depends on its mass and radius. The mass hasn’t changed, and the radius of that material has only changed by a trivial amount, because expanding a thin shell by a factor of two doesn’t move anything very far. Consequently, even though the density has changed by a factor of two, the pressure changes very little.
Finally, to obtain the change in temperature, we again assume a powerlaw equation of state $P \propto \rho^a T^b$, so that 

$$\frac{dP}{P} = a \frac{d\rho}{\rho} + b \frac{dT}{T}.$$ 

Plugging the result we obtained for $dP/P$ from hydrostatic equilibrium into this formula based on the equation of state gives 

$$\left(4 \frac{dr}{r_{sh}} - a \right) \frac{d\rho}{\rho} = b \frac{dT}{T}$$

Instability occurs when the two coefficients have opposite signs, since this means that expansion of the star, which lowers $\rho$, increases the temperature, driving more nuclear burning and more expansion. Since pressure never decreases with increasing temperature, $b$ is never negative. The coefficient on the left, however, can be: $a$ is always positive (1 for an ideal gas, $4/3$ for a degenerate relativistic gas, $5/3$ for a degenerate non-relativistic gas), and $dr/r_{sh}$ can be arbitrarily small for a sufficiently thin shell. This means that thin shells are always unstable to thermonuclear runaway.

As with the case of a degenerate star, this runaway cannot continue indefinitely. For a degenerate star, stability returns when the temperature becomes high enough to lift the degeneracy. For a thin shell, stability returns when the pressure in the shell becomes large enough to expand it to the point where it is no longer thin, and $dr/r_{sh}$ becomes larger than $a/4$.

II. Global Dynamical Stability

We have now completed our discussion of thermal instabilities, those that take place over a Kelvin-Helmholtz timescale within stars that are in hydrostatic equilibrium. We now turn to the question of dynamical instabilities, those that involve departures from hydrostatic balance. Again, we can take advantage of the fact that $t_{KH} \gg t_{dyn}$. Since $t_{KH}$ is the time required for processes involving heat or radiation, in modeling dynamical instabilities on timescales $t_{dyn}$ we can generally treat stars as adiabatic, meaning that there is negligible heat exchange. Today we will only consider global instabilities, those involving the entire star. We will leave local instabilities for the next class.

A. Stability Against Homologous Perturbations

The general theory of hydrodynamic stability is a complex one, but we can obtain the basic results by considering one particular type of perturbation: a homologous perturbation. A homologous perturbation is one in which we expand or contract the star uniformly, so that every shell expands or contracts by the same factor.

To see how a star responds to a homologous perturbation, we begin by recalling the equation of motion for a shell of material in the star, which we wrote down
early in the class. The equation is
\[ \ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho} \frac{dP}{dr}. \]

The first term on the right is the gravitational force, and the second is the pressure force. The standard equation of hydrostatic balance amounts to setting \( \ddot{r} = 0 \) in this equation. For convenience we will multiply both sides by \( dm = 4\pi r^2 \rho dr \), which gives
\[ dm \ddot{r} = -\frac{Gm}{r^2} dm - 4\pi r^2 dP, \]
where \( dP \) is the change in pressure across the shell.

Now consider a star that is initially in hydrostatic equilibrium, meaning that \( \ddot{r} = 0 \) for every shell, and add a homologous perturbation in which we expand or contract the star by a factor \( 1 + \epsilon \). That is, a shell that was previously at radius \( r_0 \) is moved to radius \( r_0(1 + \delta r/r_0) \). Expansion corresponds to \( \delta r/r_0 > 0 \), and contraction to \( \delta r/r_0 < 0 \). We assume that the perturbation is small, so \( |\delta r/r_0| \ll 1 \). This perturbation will also cause the pressure everywhere within the star to change by some factor. We let \( P_0 \) be the pressure before the perturbation, and we write the new pressure as \( P_0(1 + \delta P/P_0) \). We expect \( |\delta P/P_0| \ll 1 \) as well.

The unperturbed configuration satisfies
\[ 0 = -\frac{Gm}{r_0^2} dm - 4\pi r_0^2 dP_0. \]

Inserting the perturbed radius and pressure into the equation of motion gives
\[ dm \frac{d^2}{dt^2} \left[ r_0 \left( 1 + \frac{\delta r}{r_0} \right) \right] = -\frac{Gm}{[r_0(1 + \delta r/r_0)]^2} dm - 4\pi \left[ r_0 \left( 1 + \frac{\delta r}{r_0} \right) \right]^2 d \left[ P_0 \left( 1 + \frac{\delta P}{P_0} \right) \right]. \]

We now Taylor expand and keep only terms that are linear in \( \delta r/r_0 \) and \( \delta P/P_0 \). To remind you, the first term in the Taylor expansion of a polynomial is
\[ (1 + x)^n = 1 + nx + O(x^2). \]

Doing the expansion and plugging in gives
\[ dm \ddot{r} = -\left( 1 - 2\frac{\delta r}{r_0} \right) \frac{Gm}{r_0^2} dm - \left( 1 + 2\frac{\delta r}{r_0} + \frac{\delta P}{P_0} \right) 4\pi r_0^2 dP_0 \]
\[ dm \ddot{r} = 2\frac{Gm}{r_0^3} \delta r dm - 4\pi \left( 2\frac{\delta r}{r_0} + \frac{\delta P}{P_0} \right) r_0^2 dP_0, \]
where in the second step we cancelled the terms \(-Gm dm/r_0^2 \) and \( 4\pi r_0^2 dP_0 \), since they add up to zero.

This equation describes how the perturbation \( \delta r \) varies in time. To make progress, however, we must know how \( \delta P \) is related to \( \delta r \), and this is where we make use
of the fact that, over short timescales, the star behaves adiabatically. Recall that for an adiabatic gas, we have

$$P = K_a \rho^{\gamma_a},$$

where $\gamma_a$ is the adiabatic index, which depends only on the microphysical properties of the gas (i.e. is it degenerate or not, relativistic or not). Suppose that the perturbation causes the density to change from its original value $\rho_0$ to a new value $\rho_0(1 + \frac{\delta\rho}{\rho_0})$. Since the gas is adiabatic, the perturbed pressure and density must satisfy the same adiabatic equation of state as the unperturbed values, so

$$P_0 \left(1 + \frac{\delta P}{P_0}\right) = K_a \left[\rho_0 \left(1 + \frac{\delta\rho}{\rho_0}\right)\right]^{\gamma_a} \approx K_a \rho_0^{\gamma_a} \left(1 + \gamma_a \frac{\delta\rho}{\rho_0}\right)$$

$$\delta P = K_a \rho_0^{\gamma_a} \gamma_a \frac{\delta\rho}{\rho_0}$$

$$\frac{\delta P}{P_0} = \gamma_a \frac{\delta\rho}{\rho_0}.$$

The last step is to relate the change in density $\delta\rho$ to the change in radius $\delta r$. The mass of a shell is

$$dm = 4\pi r^2 \rho dr.$$

Homologous expansion or contraction involves changing $r_0$ to $r_0(1 + \frac{\delta r}{r_0})$, changing $dr_0$ to $dr_0(1 + \frac{\delta r}{r_0})$, and $\rho$ to $\rho(1 + \frac{\delta\rho}{\rho_0})$, while leaving the shell mass $dm$ unchanged. Thus

$$dm = 4\pi \rho_0^2 \left[ \left(1 + \frac{\delta r}{r_0}\right)^2 \rho_0 \left(1 + \frac{\delta\rho}{\rho_0}\right) dr_0 \left(1 + \frac{\delta r}{r_0}\right) \right] = 4\pi r_0^2 \rho_0 dr_0 \left(1 + 3 \frac{\delta r}{r_0} + \frac{\delta\rho}{\rho_0}\right),$$

where, again, we have linearized and dropped higher-order terms in the perturbations. However, we know that $dm = 4\pi r_0^2 \rho_0 dr_0$ exactly, so all the terms in the parentheses except the 1 must vanish. Thus we have

$$\frac{\delta\rho}{\rho_0} = -3 \frac{\delta r}{r_0}$$

Combining this with the relationship we derived from the adiabatic equation of state shows that

$$\frac{\delta P}{P_0} = -3\gamma_a \frac{\delta r}{r_0},$$

and substituting this into the perturbed equation of motion gives

$$dm \ddot{\delta r} = 2 \frac{Gm}{r_0^3} \delta r dm - 4\pi r_0^2 \left(2 \frac{\delta r}{r_0} - 3 \gamma_a \frac{\delta r}{r_0}\right) dP_0$$

$$= \left[2 \frac{Gm}{r_0^2} dm - 4\pi r_0^2 dP_0(2 - 3\gamma_a)\right] \frac{\delta r}{r_0}.$$
Recall that the unperturbed configuration satisfies
\[ \frac{Gm}{r_0^2} dm = -4\pi r_0^2 dP_0. \]

Making this substitution for \(-4\pi r_0^2 dP_0\) gives
\[ \ddot{\delta r} = -(3\gamma_a - 4) \frac{Gm}{r_0^3} \delta r. \]

This is the differential equation describing a harmonic oscillator, and it has the trivial solution
\[ \delta r = Ae^{i\omega t}, \]
where
\[ \omega = \pm \sqrt{(3\gamma_a - 4) \frac{Gm}{r_0^3}}. \]

For a non-relativistic ideal gas, \(\gamma_a = 5/3\), so the term inside the square root is positive and \(\omega\) is real. This means that \(i\omega t\) is a pure imaginary number, so \(\delta r\) varies sinusoidally in time – the response of the star to the perturbation is to oscillate at a constant amplitude \(A\). This is a stable behavior.

On the other hand, suppose we have a gas that is not a non-relativistic ideal gas, and has a different value of the adiabatic index. If \(\gamma_a < 4/3\), then the term inside the square root is negative, and \(\omega\) is an imaginary number. In this case the term \(i\omega t\) that appears in the numerator of the exponential is a real number, which can be positive or negative depending on whether we take the positive or negative square root – both are valid solutions. A negative real number corresponds to a perturbation that decays exponentially in time, which is stable.

On the other hand, a positive real number for \(i\omega\) corresponds to a solution for \(\delta r\) that grows exponentially in time. This is an instability, since it means that a small perturbation will grow to arbitrary size, or at least to the size where our analysis in the limit of small \(\delta r\) breaks down. The characteristic time required for this growth is just \(1/i\omega\). Note that \(1/(i\omega) \sim 1/\sqrt{G\rho} \sim t_{\text{dyn}}\). Thus if \(\gamma_a < 4/3\), the star will be unstable on dynamical timescale.

B. Applications of Instability

The limit that a star becomes unstable for \(\gamma_a < 4/3\) has consequences in a number of circumstances. We have already explored one: a star that is dominated by the pressure of a relativistic gas (of either non-degenerate or degenerate electrons or of photons), approaches \(\gamma_a = 4/3\). This causes stars that approach this limit to become unstable.

Another situation where a star can approach \(\gamma_a = 4/3\) is when ionization-type processes become important. Recall that we showed toward the beginning of the class that a partially ionized gas can have \(\gamma_a\) below 4/3. The dotted lines show
Clearly for $\chi/k_BT$ large enough, $\gamma_a < 4/3$ over a significant range of ionization fractions.

The centers of stars are fully ionized and their surfaces are fully neutral, so only the small parts of stars where the gas is transitioning between these are extremes are subject to ionization instability. This causes them to oscillate, but the amount of mass involved is small, and it is trapped between two stable regions. Thus the consequences are small.

However, other ionization-like mechanisms that operate at high temperatures can also produce $\gamma_a < 4/3$, and these can have more severe consequences. One such example is the photodisintegration of iron nuclei and conversion of photons into electron-positron pairs at temperatures above several times $10^9$ K. Both of these process are ionization-like in the sense that they use increases in thermal energy to create new particles rather than to make the existing particles move faster. Thus doing work on a gas in this condition does not cause its temperature to increase by any significant amount, and in doing work the temperature of the gas does not decrease much – everything is buffered by creation and destruction of particles. This is the hallmark of a gas with small $\gamma_a$.

Unlike hydrogen ionization, these process can take place at the centers of stars and can involve a significant amount of mass. This analysis suggests that, if they do take place, they compromise the stability of the star as a whole. Indeed, this is exactly what we think happens to initiate supernovae in massive stars: the core becomes hot enough that photodisintegration and/or pair creation push the adiabatic below $4/3$, initiating a dynamical instability and collapse.

III. Opacity-Driven Instabilities

We will end this class by examining one more type of instability that can occur on stars: instability driven by variations in opacity. Although we will not be able to develop an analytic theory of how these work at the level of this class, we can understand their general behavior. Given the vital importance of this sort of instability for all of
astronomy (as we will see), it is worth understanding how it works.

A. The $\kappa$ Mechanism

The mechanism for instabilities based on opacity, called the $\kappa$ mechanism since opacity is written with a $\kappa$, was worked out the mid-20th century. However, the basic idea for opacity-driven instabilities was suggested by Eddington, based on analogy with a steam valve.

Suppose there is a layer in a star that has the property that its opacity increases as it is compressed. If such a region is compressed, the increase in opacity will reduce the flow of heat through it, trapping more heat in the stellar interior. The layer acts like a valve that is closed. Closing the valve and trapping heat will raise the pressure interior to the opaque layer, causing it to expand. This expansion will decrease the opacity, opening the valve and letting the trapped heat out. This reduces the pressure in the stellar interior, reversing the expansion and letting the layer fall back. This raises its density and opacity, starting a new cycle.

Clearly this mechanism only operates if the opacity increases with density. However, this is generally not the case. Free-free opacity obeys $\kappa \propto \rho T^{-7/2}$, so

$$\frac{d\kappa}{\kappa} = \frac{d\rho}{\rho} - \frac{7}{2} \frac{dT}{T}.$$

For an adiabatic ideal gas, we have seen that

$$\frac{dP}{P} = \gamma_a \frac{d\rho}{\rho},$$

and since $P \propto \rho T$, we also know that

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}.$$

Thus for an adiabatic ideal gas, we have

$$\frac{dT}{T} = (\gamma_a - 1) \frac{d\rho}{\rho}.$$

Plugging this in, we see that for adiabatic gas, the opacity change $d\kappa$ associated with a small change in density $d\rho$ is

$$\frac{d\kappa}{\kappa} = \left( \frac{9 - 7\gamma_a}{2} \right) \frac{d\rho}{\rho}.$$

Thus $d\kappa$ and $d\rho$ have opposite signs unless $\gamma_a < 9/7 = 1.29$. Since the star is unstable only if $d\kappa$ and $d\rho$ have the same sign, i.e. increasing density increases opacity, this means that instability occurs only for $\gamma_a < 9/7$. (In fact the real condition is a bit more complicated than this, but this gives the basic idea.)
Gasses composed of relativistic and non-relativistic, degenerate and non-degenerate particles all have $\gamma_a > 4/3$, so the instability does not operate throughout most of the star. However, we have just been reminded that $\gamma_a$ can be small in the partially ionized zones of a star. In these regions of the star, this instability does operate, and these regions act like a piston, driving pulsations into the rest of the star.

Whether these pulsations actually do anything significant depends on how large the instability zone is, where it is located in the star, and how luminous the star is. There are two main instability zones, one associated with hydrogen ionization and one with helium ionization.

If the star is too hot, the ionization zones are located very close to the stellar surface, and thus they occur in a region where the density is low. This makes the piston ineffective, because it is driven by too little mass to excite motions in the rest of the star. Conversely, if a star is too cool, the ionization zones are deep in the star. The overlying layers of the star, which we will see next class are convective, then damp out the motions, and again nothing happens. Thus instability is possible only in a certain range of surface temperatures.

Moreover, since the instability is ultimately driven by the star’s radiation, so the strength with which it is driven depends on the star’s luminosity. The instability does not operate if the luminosity is too low. It turns out that it for this reason it does not generally operate in main sequence stars, because those which are luminous enough to meet the minimum luminosity condition are too hot at their surfaces, and those with cool enough surfaces are not sufficiently luminous.

Post-main sequence stars, however, can be unstable to the $\kappa$ mechanism, and this causes them to pulsate.

B. Stellar Pulsation and Variable Stars

The $\kappa$ mechanism can cause instability in stars in several different parts of the HR diagram. The regions of instability are generally characterized by a minimum luminosity and a narrow range of surface temperatures, and thus are called instability strips, since they appear as vertical strips in the diagram. The most famous of the variable stars classes is the Cepheids.

[Slide 1 – the instability strip]

Variable stars are important because the period of the oscillation depends on the luminosity of the star – which is not surprising, since the luminosity determines how hard the instability is driven. This relation was first discovered empirically in 1908 by Henrietta Swan Leavitt, and has now been understood from first principles.

[Slide 2 – light curve of SU Cygni]

[Slide 3 – Cepheid in M100 seen by HST]
The Cepheid period-luminosity relation is important in astronomy because it provides a distance indicator. Since one can compute the luminosity from the star’s observed period, one can determine its distance by comparing the observed heat flow to the luminosity. Cepheids are bright enough to be seen in other galaxies, and thus can be used to determine the distance to those galaxies. This technique was first used on a large number of galaxies by Edwin Hubble, leading to the discovery of the expansion of the universe.