Oscillators and waves crop up in all sorts of contexts, from pendula and ocean waves to electromagnetic waves and planetary librations. Generally, an oscillator requires an equilibrium state and a restoring force if it deviates from that state. Another useful way of thinking about oscillations is that they involve a periodic conversion of one kind of energy (often kinetic) to another (often potential), and back again.

The most useful characteristic of an oscillator is its period $T$, or more usually its angular frequency, $\omega = 2\pi/T$.

Part of the reason for focusing on period rather than velocity is that some (dispersive) waves have two kinds of velocities, group and phase velocities, which may differ from each other (see below).

If a physical oscillator is displaced from its equilibrium state, it will experience a restoring force. For simple harmonic motion, the restoring force is proportional to the displacement, that is $F = -kx$ where $k$ is the spring constant. This force generates an acceleration $F/m = \frac{d^2x}{dt^2}$. The result is an oscillation with a natural frequency independent of the displacement:

$$\omega^2 = \frac{k}{m}$$  \hspace{1cm} (1)

The frequency depends on the spring constant (restoring force per unit displacement) and the mass (resistance to that force). Most of the difficulty in oscillators lies in determining the correct equivalents to $k$ and $m$. For instance, in torsional systems, we need to consider the torque per unit angular displacement (equivalent of $k$), and the moment of inertia of the system (equivalent of $m$).

To derive $\omega$ another way, consider the energies involved. The maximum potential energy is $\frac{1}{2}kx^2$. The maximum kinetic energy is $\frac{1}{2}mv^2$. We need to convert from velocity to frequency, which we do by writing $v \sim x\omega$. Equating potential energy to kinetic energy, we arrive at equation (1). The energy approach is sometimes easier, but one should generally convert $v$ to $\omega$ first.

### Pendulum

For small angular displacements $\theta$, the restoring force is $mg\theta \approx mgx/l$ where $l$ is the pendulum length. So the restoring force per unit displacement is $mg/l$ and the mass is $m$, giving us

$$\omega^2 = \frac{g}{l}$$  \hspace{1cm} (2)

In terms of energy, the maximum potential energy is $mgh \approx mgl\theta^2$. The kinetic energy is $mv^2 \approx ml^2\theta^2\omega^2$, making use of the relationship $v \sim x\omega$. Again, putting these two together, we get equation (2).

For a wire of length $L$ under tension $T$ (N), the potential energy arising from a perpendicular displacement $x$ is $Tx^2/L$ (by Pythagoras) and the kinetic energy is $\sim \sigma L(x\omega)^2$, where $L$ is the length and $\sigma$ the mass per unit length. So this gives us

$$\omega^2 \sim \frac{T}{L^2\sigma}$$  \hspace{1cm} (3)

A wire under higher tension vibrates at a higher frequency, a heavier wire at a lower frequency.
Example: How much faster does a six-foot person cover the ground than a five-foot person?

Example: Planets and satellites wobble backwards and forwards slightly (librations). The tidal torque per unit angular displacement of a satellite is given by $\sim GM_p\Delta I/a^3$ where $a$ is the semi-major axis, $G$ is the gravitational constant, $M_p$ is the primary mass and $\Delta I$ is the moment of inertia difference for the satellite (a sphere would experience no torques). How does the libration frequency compare with the orbital frequency?

Damped oscillators

Real systems invariably involve some kind of damping, often proportional to the velocity. This is typically introduced as an extra force, $-f\frac{dx}{dt}$, where $f$ is the damping coefficient, and the solution is developed using complex numbers. Damping changes the natural frequency (slightly), introduces a phase lag, and causes the amplitude to decay to zero. The decay rate depends on the material properties of the system.

The easiest way to treat damping is to think about energy. Since power = force $\times$ velocity, the rate of energy loss is $f \left(\frac{dx}{dt}\right)^2 \sim f x^2 \omega^2$. By comparing this loss rate with the total potential energy, we obtain the damping timescale. For instance, for our standard oscillator we obtain

$$t_{damp} \sim \frac{k x^2}{f x^2 \omega^2} = \frac{k}{f \omega^2}$$

A more common way of expressing this is the dissipation factor $Q$ which is the number of cycles it takes for the energy to decrease. This is just $t_{damp} \omega$ so we end up with

$$Q \sim \frac{k}{f \omega} = \sqrt{\frac{k m}{f}}$$

A smaller $Q$ indicates more damping (i.e. larger $f$). $Q$ varies widely, but is often of order 100.

Example: What’s the $Q$ of a dropped tennis ball? Of water in a bathtub? Of the Earth?

Continuous materials

For continuous materials, it is often useful to replace the spring constant $k$ with $El$, where $E$ is Young’s modulus or some other measure of rigidity having units of pressure and $l$ is a lengthscale (for waves, this is usually the wavelength).

We have

$$\omega^2 \sim \frac{El}{m} \sim \frac{1}{l^2 \rho} \tag{4}$$

where $\rho$ is the density. [For a beam of length $l$ and with $w$, we replace $l^2$ with $l^4/w^2$.]

Example: What is the lowest frequency of normal mode oscillations for the Earth?

[Answer: Let’s take $l \sim R$, the radius of the Earth. The density is about 5000 kg m$^{-3}$ and the Young’s modulus is about $10^{11}$ Pa (from Week 1). That gives us about 0.001 rad/s, or a period of a few hours.]

Example: How rapidly does a skyscraper vibrate?

[Answer. Let’s use the beam example, with $l=300$ m (say) and $w=50$ m. Because it’s not a uniform material, the density and rigidity may be different. The density will certainly be lower, say 300 kg m$^{-3}$. If we just assume a rigidity of 100 GPa we would get an oscillation period that is
absurdly fast. We might choose to scale the rigidity by the area of the load-bearing material, which is probably only a few percent of the total area. That gives an oscillation period of a few seconds. That’s about right.]

Example Derive an equivalent expression to (4) for a fluid body. What is the lowest frequency of normal mode oscillations for the Sun? [This is most easily done using energy balance, but to get the potential energy you have to remember to integrate the displacement].

[Answer: From the waves section, the energy required to build a hemisphere-scale wave of amplitude $h$ is $\sim \rho h^2 R^2 g$. The velocity $v \sim h \omega$ and if we assume that the velocity is felt throughout the body, the kinetic energy is then $\sim \rho R^3 v^2$. So we get $\omega^2 \sim g/R$ or equivalently $\omega^2 \sim G\rho$. So for the Sun we’ll assume a density similar to that of water get a period of about 7 hours (not very different from the period for an elastic Earth).]

Waves

A wave is a propagating oscillation. For a particular frequency of wave, the propagation velocity of that wave is called the phase velocity and is given by

$$v_{ph} = \frac{\omega}{k} = f\lambda$$

(5)

where $f$ is the frequency (in Hz), $k = 2\pi/\lambda$ is the wavenumber and $\lambda$ is the wavelength. [Note that this $k$ is different from the spring constant!]

If the wave packet consists of waves of more than one frequency, then the propagation velocity of the packet of waves, the group velocity, is given by

$$v_{gr} = \frac{\partial \omega}{\partial k}$$

If waves are dispersive, then the group velocity and the phase velocity are not the same thing, and the phase velocity is wavelength-dependent. Sanjoy has a lucid discussion of this in his chapter on Waves. Inertial waves (such as Rossby waves) are a class of waves in which the restoring force is provided by the Coriolis effect. For these waves, the group velocity and phase velocity are perpendicular to each other. This makes my head hurt.

If we take equation (4) and assume that $l \sim \lambda$, then we can use equation (5) to derive the (phase) velocity for sound waves in an elastic solid:

$$v_{ph}^2 \sim \frac{E}{\rho}$$

This expression can also be arrived at by remembering that any quantity with units of pressure (like $E$) can also be thought of as energy per unit volume, while $v^2$ is just kinetic energy per unit mass, and equating the two energies.

So in theory seismic wave velocities are frequency-independent i.e. non-dispersive. (In practice surface waves are dispersive, because different wavelengths sample to different depths, and both $E$ and $\rho$ vary with depth).

Example How does this (macroscopic) definition of sound waves relate to an atomistic description of sound transport?

[Answer: Recall from Week 1 that $E \sim E_c/a^3$, where $E_c$ is the binding energy and $a$ is the atomic separation. Also $\rho \sim m/a^3$ where $m$ is the mass of an atom/molecule. So we get $v^2 \sim E_c/m$, which is equivalent to setting the kinetic energy equal to the binding energy.]
Example: What is the equivalent expression for sound waves in air? Again, how do we relate this answer to an atomistic description?

[Answer: One way is to recognize that for a gas, the equivalent of $E$ is the pressure, $P$. So for sound waves we get $v^2 \sim P/\rho \sim RT/\mu$, where $\mu$ is the molar mass. We could equally write this as $v^2 \sim kT/m$, where $k$ is Boltzmann’s constant, which is the atom-scale equivalent and means we have set the kinetic energy equal to the thermal energy.]

Example: The energy stored per unit volume in a magnetic field is $\sim B^2/\mu_0$, where $B$ is the field and $\mu_0$ the permeability of free space. Magnetic field lines have an effective tension, which results in transverse propagating waves (Alfven waves). For a plasma of density $\rho$, what is the Alfven wave velocity?

[Answer: The energy stored per unit volume is a potential energy. The equivalent kinetic energy per unit volume is $\sim \rho v^2$. So setting the two equal we obtain $v^2 \sim B^2/\mu_0\rho$.]

Water Waves

In the deep water limit, $\lambda \ll D$ where $D$ is the ocean depth and the waves do not feel the influence of the seafloor. The column of water responding to the surface disturbance has a thickness $\approx \lambda$.

Moving the water from a flat state to waves with an amplitude $h$ involves a potential energy per wavelength $\sim \lambda h^2 g \rho$ per unit width.

In treating the kinetic energy, we have to be careful with velocity. The upwards velocity is $\omega h$ and extends to a depth $\lambda$, and is comparable to the horizontal velocity. So the kinetic energy per wavelength per unit width is $\sim \rho h^2 \lambda^2 \omega^2$. Equating the two terms we obtain

$$\omega^2 \sim \frac{g}{\lambda} \sim gk$$

which can also be rearranged using equation (5) to give us $v_{ph}^2 \sim g \lambda$. This latter result shows that deep water waves are dispersive - the velocity depends on the wavelength.

Example: A conventional ship sets up a bow wave which is roughly equal to the length of the ship. What is the maximum speed of a small motor boat, and what is the Froude number at this speed? How might this limit be avoided?

[Answer: We’ll assume that the maximum speed is given by a Froude number $\sim 1$, so that $v^2 \sim gL$. Taking $L \approx 10$ m, we get a maximum speed of 10 m/s, about right. Boats can go faster than this, but doing so requires very powerful engines. An alternative is to raise the hull above the water i.e. a hydrofoil.]

In the shallow water approximation, where $\lambda \gg D$, the waves do feel the influence of the seafloor. The potential energy expression is the same as before, but now the horizontal velocity is bigger than the vertical velocity by a factor of $\lambda/D$ (why?). The kinetic energy per unit width is now $\sim \rho h^2 \lambda^3 \omega^2 / D$. This gives us

$$\omega^2 \sim \frac{gD}{\lambda^2}$$

and thus $v_{ph}^2 \sim gD$, so that shallow water waves are non-dispersive. This implies that wave velocity decreases as the ocean shallows; by continuity, the wave amplitude must increase, until at some point the wave becomes over-steepened and breaks.

Example: A tsunami with a 5 km wavelength is a shallow-water wave, even in the central Pacific ocean. How fast does it travel? What does conservation of mass imply about the behaviour of this wave as it approaches the shore?
[Answer: \( v \sim (gD)^{1/2} \) gives about 200 m/s, very fast indeed. As \( D \) decreases, so does \( v \), which means that the water starts to pile up. The amplitude of tsunamis in the deep ocean is very small, but they can still be very destructive as they reach the shore.]

Similar results hold if surface tension \( \gamma \) rather than gravity is the dominant restoring force. The extra energy per unit width required to take a piece of water from flat to sinusoidal is \( \sim \gamma h^2/\lambda \), by Pythagoras. Equating this potential energy to kinetic energy as before, we obtain for deep water

\[
\omega^2 \sim \frac{\gamma}{\lambda^3 \rho}
\]

**Example** What is the period of surface-tension driven ripples in shallow water?

[Answer: For shallow water the kinetic energy per unit width is \( \sim \rho h^2 \lambda^3 \omega^2/D \). Equating this to the potential energy we obtain \( \omega^2 \sim \gamma D / \rho \lambda^3 \).]

**Example** What is the characteristic frequency of a bubble expanding and contracting underwater (cavitation)?

**Example** One can imagine planetary oscillations in which contracting material releases gravitational energy, causing heating and expansion. What determines the period of these oscillations?

**Example** What is the Brunt-Vaisala frequency of a fluid? This effect occurs when a stably stratified fluid is perturbed from an initial density gradient \( d\rho/dz \).

[Answer: We can do this one by dimensional analysis. Presumably the frequency increases with increasing \( d\rho/dz \) and increasing \( g \). To make the dimensions work we can put \( \rho \) in the denominator, which means that the frequency depends on the fractional density gradient, which makes intuitive sense. So we end up with \( \omega^2 \sim g \rho / a^2 \).]

**Gravitational Waves**

These have just been detected! The propagation velocity is \( c \) and the frequency is set by the characteristics of the system (e.g. orbital period of a black hole binary). The interesting quantity is the anisotropic strain generated, which is what has been measured.

Although it is not obvious to me why, the strain amplitude depends on three lengthscales: the black hole size \( R = GM/c^2 \), the binary separation \( a \) and the distance to the observer \( r \). The resulting amplitude \( \varepsilon \) is

\[
\varepsilon \sim \frac{R^2}{ra}
\]

As the final merger takes place, \( a \approx R \) and so \( \varepsilon \sim R/r \). For a 10 km size black hole at \( 10^9 \) LY (=10^{25} m) the strain is \( 10^{-21} \), a very small (but measurable) quantity. The LIGO detector is about 3 km long, so the measured displacement would be \( 3 \times 10^{-18} \) m or a few thousandths of the radius of a single proton (!).