Introduction
So far, we have not really had to memorize any equations. In studying heat transfer, it is actually helpful to learn one or two. This is because an equation acts as a mnemonic: it contains a lot of information, which otherwise you would have to remember piecemeal.

To deal with equations, you need to learn three important skills, examples of which you will see below:

(i) Identifying the processes represented
(ii) Approximating differential terms
(iii) Balancing the leading (most important) terms

Heat Production
An important macroscopic property is the specific heat capacity \( C_p \), with units of \( \text{J kg}^{-1} \text{K}^{-1} \), which tells you how much energy it takes to heat or cool a substance. At an atomic level, it must obviously be related to Boltzmann’s constant \( k \). For gases \( C_p \sim 3R \), where \( R \) is the gas constant \( (= k \times N_A, \approx 10 \text{ J mol}^{-1} \text{K}^{-1}) \). For solids the numbers are generally similar: \( 3R \) gives \( 10^3 \text{ J kg}^{-1} \text{K}^{-1} \) for an atomic mass of 30, which is about right.

The change in temperature \( \Delta T \) of a substance depends on its specific heat capacity \( C_p \) and the amount of energy added/subtracted \( \Delta E \):

\[
\Delta E = mC_p\Delta T
\]

where \( m \) is the mass of the substance. If a phase change is involved, an additional energy input \( mL \) is required, where \( L \) is the latent heat \( (\text{J kg}^{-1}) \). Recall that the latent heat can be derived by considering the energy required to break bonds; a binding energy of 1 eV translates into a latent heat (of vapourization) of about 3 MJ/kg.

Example An adiabatic gas cools as it rises, because the pressure drops and it therefore expands. Use potential energy to determine the lapse rate \( \Delta T/\Delta z \) of a perfect gas. What happens if condensation occurs as the gas rises?

If we consider a system in which no heat is lost, then if the rate of heat production \( (\text{in W m}^{-3}) \) is \( H \), equation (1) may be written as

\[
\rho C_p \frac{\partial T}{\partial t} = H
\]

where \( \rho \) is the density. Higher rates of heating (RHS) result in faster temperature changes (LHS).

Example What’s the heating rate \( (\text{in W/kg}) \) that a microwave oven can generate?
Example Estimate \( H \) \( (\text{in W kg}^{-1}) \) for a human being and the Sun.

Heat Transfer
In reality, heat production is usually balanced by some kind of heat transfer. Heat transfer takes place when spatial variations in temperature occur, and happens via conduction, radiation or advection. A special form of advection is convection, where the fluid motion is itself driven by temperature differences.
Conduction

For any diffusive process, the flux depends on the concentration gradient and a constant of proportionality. This is Fick’s law and for heat flux may be expressed as:

\[ F = -k \nabla T \]  \hspace{1cm} (3)

where \( F \) is the heat flux (W m\(^{-2}\)), \( k \) is the thermal conductivity (W m\(^{-1}\) K\(^{-1}\)) and \( T \) is the temperature. NB “flux” means per unit area (W m\(^{-2}\)); “flow” is the total power (W).

The thermal conductivity is directly related to the thermal diffusivity \( \kappa \) by \( k = \kappa \rho C_p \). The thermal diffusivity for non-metals depends on the mean free path and velocity (see Week 2). For gases \( \kappa \sim 3 \times 10^{-5} \) m\(^2\) s\(^{-1}\), so a typical thermal conductivity for a gas is \( \sim 3 \times 10^{-2} \) W m\(^{-1}\) K\(^{-1}\).

Solids and liquids have higher (vibration) velocities (by one order of mag), but smaller mean free paths (by three orders of mag). So we would expect the diffusivity of solids and liquids to be roughly \( 10^{-6} \) m\(^2\) s\(^{-1}\), which is about right. For solids and liquids, \( C_p \) (per mole) is similar to that of gases, but \( \kappa \) is two OOM smaller and \( \rho \) is three OOM bigger. So we would expect thermal conductivity of solids to be \( \sim 0.3 \) W m\(^{-1}\) K\(^{-1}\) (real values are a bit higher than this).

Example What’s the rate of heat production inside the Earth (W/kg)?

Thermal conductivity in metals is higher. Why? Because the electrons are another way of carrying heat, and electrons travel \( \sim 300 \) times faster than atoms, and have a longer mean free path. On the other hand, each electron has a smaller heat capacity than each atom. It turns out that metals have a conductivity of \( \sim 100 \) W m\(^{-1}\) K\(^{-1}\).

Combining equations (1) and (3) and considering the heat flows within, into and out of a small box, we can derive a more general equation

\[ \rho C_p \frac{\partial T}{\partial t} = k \nabla^2 T + H \]  \hspace{1cm} (4)

The first term on the RHS is the effect of conductive heat flow, the second is due to heat production.

Equation (3) involves a first derivative, but equation (4) involves a second derivative - how come? The answer is that for a fixed volume in space the heat flux into the volume depends on equation (3), and so does the heat flux out. So whether the volume heats up or cools down depends on the difference between the heat fluxes in and out. For a small enough volume, the difference between two heat fluxes becomes the gradient (first derivative) of the heat flux, which is the second derivative of the temperature.

Approximating

Equation (4) gives the relationship between cooling rate, heat flow and heat production. We can approximate the importance of each of these terms by getting rid of all the nasty differential signs:

\[ \frac{\partial T}{\partial t} \sim \frac{T}{t}, \quad \nabla T \sim \frac{T}{d}, \quad \nabla^2 T \sim \frac{T}{d^2}, \ldots \]

where it is now understood that \( T, t, d, \ldots \) represent characteristic temperature-, time- and length-scales for the particular problem that we are interested in.

Having made these simplifications, equation (4) ends up as

\[ \rho C_p \frac{T}{t} \sim k \frac{T}{d^2} + H \]  \hspace{1cm} (5)
and we can now look at situations in which one of these terms can be neglected.

If we are in steady state, then the LHS of equation (5) is zero and heat production is balanced by heat conduction. We obtain

$$T \sim \frac{Hd^2}{k}$$

where here $T$ really represents the temperature difference driving conduction (this explains the disappearance of the minus sign!). Higher heat production and lower thermal conductivities lead to higher internal temperatures.

*Example* What’s the internal temperature of an elephant? What do you conclude?

*Example* What’s the internal temperature of the Earth? Do you believe the answer?

If heat production is negligible, we balance heat conduction against temperature change in equation (5) and obtain

$$t \sim \frac{d^2}{\kappa}$$

(6)

This gives the time it takes a temperature change to propagate a given distance $d$ in a conductive medium. An identical expression can be found by considering the heat energy contained in a box of given dimensions, divided by the characteristic heat flow.

Equation (6) is extraordinarily important, because it applies to any diffusive process: chemical, magnetic, viscous, you name it. If you know the diffusivity, you can derive the diffusion timescale from a known lengthscale, or vice versa.

*Example* Estimate the thermal conductivity of metal by e.g. considering how fast a teaspoon heats up.

*Example* How long does it take the Moon to cool?

*Example* By considering a periodic surface heat flux $F$, demonstrate that the near-surface temperature change depends on a quantity $(k\rho C_p)^{1/2}$, known as the thermal inertia.

**Advection**

Heat can also be transported by motion. The heat flux associated with an advecting fluid is

$$F = u\rho C_p \Delta T$$

(7)

where $\Delta T$ is the temperature contrast between the fluid and the surroundings. We can use this to add an extra term to equation (4):

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = \kappa \nabla^2 T + \frac{H}{\rho C_p}$$

(8)

where $u$ is the velocity. You can understand why the advection term depends on the gradient of $T$ if you think about contour lines (e.g. of temperature) being carried past you.

*Example* How much power is carried by the Gulf Stream? How does this compare with US power usage?

Balancing advection against conduction in equation (8) we obtain

$$\frac{uT}{d} \sim \frac{\kappa T}{d^2}$$
which gives us a dimensionless number \( ud/\kappa \) (known as the Peclet number). This tells us whether advection or conduction is more important.

**Example** In our incense experiment, what was the Peclet number?

**Example** How deep beneath the surface is an undisturbed geotherm reached if the surface is eroding at 10 mm/a?

**Note:** Although a naive examination of the Peclet number for e.g. the Earth’s mantle would suggest that conduction is completely unimportant compared with advection, in fact it cannot be neglected. This is because boundary layers develop, across which the heat flow is conductive (velocities are small). We will deal with **convection**, and the development of convective boundary layers, in a subsequent lecture.

Balancing advection against heat production in equation (7) we obtain

\[
T \sim \frac{Hd}{\rho C_p u}
\]

**Example** Back to our elephant: what is the flow rate of fluid (blood) transporting heat? You’ll have to correct for the small area occupied by blood vessels.

**Radiation**

Energy can be carried by photons. The black body heat flux emitted by a body at temperature \( T \) (in K) is given by

\[
F = \varepsilon \sigma T^4
\]

where \( \varepsilon \) is the emissivity (\( \approx 1 \)) and \( \sigma \) is Stefan’s constant. If you can’t remember Stefan’s constant (\( \approx 6 \times 10^{-8} \) W m\(^{-2}\) K\(^{-4}\)), you can calculate it by knowing the Sun’s temperature (\( \approx 6000 \) K) and that the Earth receives \( \sim 10^3 \) W/m\(^2\) at a distance of 200 solar radii. Or by knowing the surface temperature of the Earth and the solar constant.

The wavelength at which the maximum radiated power occurs goes as \( 1/T \). If you remember that the peak of Sun’s spectrum is in the visible (roughly 0.5 microns) and the solar photospheric temperature, then you can work out the constant of proportionality. The fraction of incoming photon energy absorbed is given by \( 1 - A \), where \( A \) is the albedo (1=perfectly reflective).

On Earth, the incoming radiation is short-wavelength (visible) but the outgoing radiation is infra-red (because the Earth’s surface is cool). This difference in wavelengths is what allows the greenhouse effect to occur.

**Example** How much of the Earth’s surface would we need to paint white to offset a 2 K temperature increase due to the greenhouse effect?

Radiative transfer through atmospheres is a difficult subject (at least for me). But there is one very useful approximation, which relates the radiating (effective) temperature \( T_e \) at the top of the atmosphere to the temperature at greater depth, for an atmosphere heated from below:

\[
T(z) \approx T_e (1 + \tau(z))^{1/4}
\]

where \( \tau \) is a dimensionless quantity called the **optical depth** which tells you how many \( e \)-folding times the incident radiation has been reduced by over a given distance. You can calculate \( \tau \) using \( \int \rho \alpha dz \) where \( \rho \) is the bulk density and \( \alpha \) is a so-called absorption coefficient (units kg\(^{-1}\) m\(^2\), roughly
$10^{-2} - 10^{-5}$ for gases). For solid objects $\alpha^{-1} \sim r \rho_s$ where $\rho_s$ is the solid density and $r$ is the particle size.

Note that below some depth ($\tau \gg 1$) atmospheres become too opaque to transmit heat efficiently by radiation, and they instead convect.

It is often possible to write radiative heat transfer through the atmosphere as a *conduction* problem (equation 3), which makes life easier. This is because a packet of atmosphere radiates upwards and downwards; there is net upwards transport of heat only if there is a temperature gradient. In this case the thermal conductivity $k$ depends on the opacity and the local temperature.

**Example** In the absence of a greenhouse effect, the surface temperature of the Earth would be about 250 K. What is the optical depth of the Earth’s atmosphere? What would you predict to be the size of the greenhouse effect on Mars?

**Useful numbers**

- $\kappa \approx 10^{-6} \text{ m}^2 \text{s}^{-1}$ (rock, ice) $10^{-4} \text{ m}^2 \text{s}^{-1}$ (gases)
- $\rho C_p \approx 3 \text{ MJ m}^{-3} \text{ K}^{-1}$ (solids) $3 \rho R$ (gases, per mole)
- $L_{\text{vap}} \approx 1 \text{ MJ/kg}$, $L_{\text{fus}} \approx 0.1 \text{ MJ/kg}$
- $\sigma \approx 6 \times 10^{-8}$ $\text{ W m}^{-2} \text{ K}^{-4}$