Reorientation of planets with lithospheres: The effect of elastic energy

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Abstract
It is commonly assumed that internal energy dissipation will ultimately drive planets to principal axis rotation, i.e., where the rotation vector is aligned with the maximum principle axis, since this situation corresponds to the minimum rotational energy state. This assumption simplifies long-term true polar wander (TPW) studies since the rotation pole can then be found by diagonalizing the appropriate (non-equilibrium) inertia tensor. We show that for planets with elastic lithospheres the minimum energy state does not correspond to principal axis rotation. As the planet undergoes reorientation elastic energy is stored in the deforming lithosphere, and the state of minimum total energy is achieved before principal axis rotation. We find solutions for the TPW of planets that include this effect by calculating the elastic stresses associated with deformation, and then minimizing the total (rotational and elastic) energy. These expressions indicate that the stored elastic energy acts to reduce the effective size of the driving load (relative to predictions which do not include this energy term). Our derivation also yields expressions for the TPW-induced stress field that generalizes several earlier results. As an illustration of the new theory, we consider TPW driven by the development of the Tharsis volcanic province on Mars. Once the size of the Tharsis load and the Mars model is specified, the extended theory yields a more limited range on the possible TPW.

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1. Introduction

True polar wander (TPW) refers to the reorientation of the rotation pole of a planet in response to changes in the inertia tensor associated with mass redistribution. The lowest kinetic energy for a rigid spinning body corresponds to rotation about the principal axis associated with the largest moment of inertia (hereafter referred to as the maximum principal axis). Hence, it is generally assumed that internal energy dissipation will ultimately drive planets to this principal axis rotational state.

Gold (1955) assumed that the rotational bulge eventually relaxes to any new position of the rotation pole. In this case the orientation of the maximum principal axis is ultimately determined by the non-hydrostatic inertia tensor (e.g., due to surface mass or internal loading) since the planet would not have any long-term memory of a previous rotational state. Goldreich and Toomre (1969) demonstrated that if the non-hydrostatic inertia tensor changes slowly (relative to the nutation frequencies of the planet), the solid angle between the maximum principal axis and the rotation pole is an adiabatic invariant. It then follows that if a planet achieves the maximum principal axis rotational state, it will continue to do so, and thus slow changes in the inertia tensor drive TPW as the maximum principal axis migrates. Willemann (1984) noted that planets with elastic lithospheres will preserve a ‘remnant rotational bulge’; that is, the development of a lithosphere on a rotating form will, in contrast to the case considered by Gold (1955), establish a memory of prior pole positions. The remnant bulge will tend to stabilize the planet since it acts to counter the driving force of any uncompensated loading, and Willemann (1984) found TPW solutions by diagonalizing an inertia tensor which includes this effect (see Matsuyama et al., 2006). However, Ojakangas and Stevenson (1989) pointed out that the minimum total energy state for a planet with an elastic lithosphere may not correspond...
to maximum principal axis rotation because reorientation generates elastic strains within the lithosphere. These strains reduce the energy available to drive further reorientation.

The goal of this paper is to extend Willemann (1984) and Matsuyama et al. (2006) to find TPW solutions that include the effect of the elastic energy stored in the lithosphere. The paper is organized as follows. Section 2 focuses on inertia tensor perturbations associated with the direct and deformational effects of surface mass loads and the deformation driven by changes in the centrifugal potential. Section 3 compares the TPW solutions derived via diagonalization of the inertia tensor and rotational energy minimization for the case where we ignore the elastic energy within the lithosphere. Section 4 outlines TPW solutions obtained by finding the minimum total energy state that includes the elastic energy in the lithosphere and compares these solutions to those in Section 3. Finally, Section 5 considers the implications of the new results for estimates of the reorientation of Mars driven by the rise of the Tharsis volcanic province.

2. Inertia tensor perturbations

The inertia tensor of a planet is given by

\[ I_{ij} = \int_V dV \rho (r^2 \delta_{ij} - r_i r_j), \]

where \( V \) denotes volume and \( \delta_{ij} \) is the Kronecker delta. We assume that in the unperturbed state the planet has a spherically symmetric (i.e., radially varying) density profile, \( \rho_0(r) \). The inertia tensor of the unperturbed planet can be written as

\[ I_{ij} = \frac{8\pi}{3} \delta_{ij} \int_0^a dr r^4 \rho_0 = I_0 \delta_{ij}, \]

where \( a \) is the mean radius of the planet.

We can expand the density perturbations, \( \rho \), relative to the unperturbed state in terms of spherical harmonic basis functions \( Y_{lm} \) as

\[ \rho(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \rho_{lm}(r) Y_{lm}(\theta, \phi), \]

where \( \rho_{lm} \) are the spherical harmonic coefficients. We adopt the normalization

\[ \int_S Y_{lm}^* (\theta, \phi) Y_{lm} (\theta, \phi) dS = 4\pi \delta_{ll} \delta_{mm}, \]

where \( S \) denotes the complete solid angle and \( \dagger \) represents the complex conjugate.

It is useful to consider the gravitational potential perturbation associated with the density perturbation. The gravitational potential external to the planet due to a density perturbation can be expanded in spherical harmonics as

\[ G(r, \theta, \phi) = \frac{GM}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \frac{a}{r} \right)^l \mathcal{G}_{lm} Y_{lm}(\theta, \phi), \]

where \( G \) is the gravitational constant, \( \mathcal{G}_{lm} \) are dimensionless spherical harmonic coefficients, and \( M \) is the total mass of the planet. The potential coefficients \( \mathcal{G}_{lm} \) are then related to the density coefficients by

\[ \mathcal{G}_{lm} = \frac{1}{(2l+1) M a^l} \int_0^a dr r^{2-l} \rho_{lm}(r) \]

(e.g., Kaula, 1968, p. 67, with suitable normalization). Note that \( \mathcal{G}_{00} = 0 \) if the total mass of the planet is conserved.

We will consider the special case of a gravitational potential perturbation with azimuthal symmetry. We denote the dimensionless coefficients for this gravitational potential, if its symmetry axis is at the north pole, as \( \mathcal{G}'_{0l} \). We can use the addition theorem for spherical harmonics to write the coefficients for the case where the symmetry axis is at an arbitrary position, given by colatitude \( \theta_G \) and longitude \( \phi_G \), as

\[ \mathcal{G}_{lm} = \frac{\mathcal{G}'_{0l}}{\sqrt{2l+1}} Y_{lm}^*(\theta_G, \phi_G). \]

Using Eqs. (1), (2), (5), and (6), and the orthogonality of the spherical harmonics [Eq. (3)], yields the following expression for the associated inertia tensor perturbations:

\[ I_{ij} = I_{00} \delta_{ij} + \sqrt{5} M a^2 \mathcal{G}'_{0l} \left( \frac{1}{3} \delta_{ij} - \epsilon_i^G \epsilon_j^G \right), \]

where

\[ I_{00} = \frac{8\pi}{3} \int_0^a dr r^4 \rho_{00}. \]

and \( \epsilon_i^G = (\sin \theta_G \cos \phi_G, \sin \theta_G \sin \phi_G, \cos \theta_G) \) is a unit vector aligned with the symmetry axis of the gravitational potential perturbation. Equation (7) allows us to obtain expressions for inertia tensor perturbations associated with any axisymmetric potential perturbation.

2.1. Surface load contributions

We begin by considering inertia tensor perturbations associated with the direct and deformational effects of a surface mass load. Once again, we consider the special case of a load with azimuthal symmetry. This assumption will enable us to derive analytic solutions, thus providing greater insight into the physics of the associated TPW, and will also be useful for comparison with previous work (specifically Willemann, 1984; Matsuyama et al., 2006). We denote the degree two coefficients for the surface mass density of this load, when centered on the north pole, as \( L'_{20} \).

The degree two potential coefficient associated with the direct effect of this surface load is

\[ \mathcal{G}'_{20} = \frac{4\pi a^2}{5M} L'_{20}, \]

where we have used Eq. (5). Using Love number theory, we can express the total degree two potential coefficient associated
with the direct and deformational effects of the load as
\[ \mathcal{G}_{20}^{L,LD'} = (1 + k_2^L) \mathcal{G}_{20}, \tag{10} \]
where \( k_2^L \) is the degree two fluid load Love number which describes the long-term response of the planet to a load forcing. Since we are interested in the long-term inertia tensor perturbations, we are implicitly assuming that all viscous stresses are presumed to have relaxed by using the fluid (or fluid limit) load Love number.

Finally, we make use of Eq. (7) to derive the inertia tensor perturbation associated with the direct and deformational effects of an axisymmetric load when the symmetry axis is at an arbitrary position with colatitude \( \theta_L \) and longitude \( \phi_L \):
\[ I_{ij}^L = \sqrt{5} M a^2 \mathcal{G}_{20}^{L,LD'} \left( \frac{1}{3} \delta_{ij} - \epsilon_i^L \epsilon_j^L \right). \tag{11} \]
In this expression, \( \epsilon_i^L = (\sin \theta_L \cos \phi_L, \sin \theta_L \sin \phi_L, \cos \theta_L) \) is a unit vector aligned with the symmetry axis of the load. We ignore the first term on the right-hand side of Eq. (7) since we assume that the mass of the planet is conserved in creating the load and thus \( L_{00} = 0 \).

### 2.2. Rotational deformation

The equilibrium figure of a rotating planet is not spherical since the planet deforms in response to the centrifugal potential. The spherical harmonic coefficients for the centrifugal potential, if the rotation pole is at the north pole, and using the normalization (3), are
\[ \mathcal{G}_{00}^R = \frac{a^3 \Omega^2}{3 G M} \tag{12} \]
and
\[ \mathcal{G}_{20}^R = \frac{-1}{3} \frac{a^3 \Omega^2}{\sqrt{5} G M}, \tag{13} \]
where \( \Omega \) is the magnitude of the rotation vector. Using Love number theory, the long-term degree two response driven by the forcing potential can be written as
\[ \mathcal{G}_{20}^{RD'} = k_2^T \mathcal{G}_{20}^R, \tag{14} \]
where \( k_2^T \) is the degree two fluid tidal Love number. Using Eqs. (7), (13), and (14), the inertia tensor perturbation associated with the rotational deformation for a rotation pole with spherical coordinates \((\theta_R, \phi_R)\) is given by
\[ I_{ij}^R = I_{00}^R \delta_{ij} - \frac{a^5 \Omega^2}{3 G} k_2^{T*} \left( \frac{1}{3} \delta_{ij} - \epsilon_i^{T*} \epsilon_j^{T*} \right). \tag{15} \]
In this expression, \( I_{00}^R \) is a spherically symmetric perturbation associated with the planetary response to the radially-directed component of the centrifugal force (i.e., the centrifugal force component associated with \( \mathcal{G}_{00}^R \)), and \( \epsilon_i^T = (\sin \theta_R \cos \phi_R, \sin \theta_R \sin \phi_R, \cos \theta_R) \) is a unit vector in the direction of the rotation pole.

Willemann (1984) considered a scenario in which the initial figure of a planet is established early in the planet’s history, during a time when the effective elastic lithospheric thickness is zero. As the planet cools the lithosphere becomes elastic; thus, the lithosphere preserves a record of the initial rotational figure in the form of a remnant rotational bulge. If the rotation pole moves or the magnitude of the rotation vector changes, the lithosphere is strained from its equilibrium shape and resists re-orientation. If we denote the magnitude of the initial (before the lithosphere forms) and final (after reorientation) rotation vectors as \( \Omega^* \) and \( \Omega \), with spherical coordinates \((\theta_R^*, \phi_R)\) and \((\theta_R, \phi_R)\) for the rotation poles respectively, then the inertia tensor perturbations associated with the change in the centrifugal potential can be written as
\[ I_{ij}^R = I_{00}^{RD} \delta_{ij} - \frac{a^5 \Omega^2}{3 G} k_2^L \left( \frac{1}{3} \delta_{ij} - \epsilon_i^{L*} \epsilon_j^{L*} \right) \]
\[ - \frac{a^5 \Omega^2}{3 G} k_2^T \left[ \Omega^2 \left( \frac{1}{3} \delta_{ij} - \epsilon_i^{T*} \epsilon_j^{T*} \right) \right], \tag{16} \]
where \( k_2^{L*} \) and \( k_2^T \) are the degree two fluid tidal Love numbers for the planet without and with an elastic lithosphere respectively. The second term on the right-hand side of Eq. (16) corresponds to the initial hydrostatic planetary figure, while the third term is the perturbation associated with the planetary response to the change in the centrifugal potential. It will be useful to rewrite Eq. (16) as
\[ I_{ij}^R = I_{00}^{RD} \delta_{ij} - \frac{a^5 \Omega^2}{3 G} (k_2^{L*} - k_2^T) \left( \frac{1}{3} \delta_{ij} - \epsilon_i^{L*} \epsilon_j^{L*} \right) \]
\[ - \frac{a^5 \Omega^2}{3 G} k_2^T \left( \frac{1}{3} \delta_{ij} - \epsilon_i^{T*} \epsilon_j^{T*} \right) \], \tag{17} \]
where we can then identify the second term on the right-hand side of Eq. (17) as the remnant rotational bulge discussed by Willemann (1984) and Matsuyama et al. (2006). Consider the special case of a planet in which an elastic lithosphere does not form. In this case, \( k_2^{L*} = k_2^T \) and Eqs. (16) and (17) simplify to Eq. (15), as would be expected.

### 2.3. Total inertia tensor perturbations

We denote the total inertia tensor perturbation due to the direct and deformational effects of the surface load and the rotational deformation associated with a change in the centrifugal potential as \( I_{ij} \equiv I_{ij}^{RD} + I_{ij}^L \). Using Eqs. (11) and (17) yields
\[ I_{ij} = I_{00} \delta_{ij} + \sqrt{5} M a^2 \mathcal{G}_{20}^{L,LD'} \left( \frac{1}{3} \delta_{ij} - \epsilon_i^L \epsilon_j^L \right) \]
\[ - \frac{a^5 \Omega^2}{3 G} (k_2^{L*} - k_2^T) \left( \frac{1}{3} \delta_{ij} - \epsilon_i^{L*} \epsilon_j^{L*} \right) \]
\[ - \frac{a^5 \Omega^2}{3 G} k_2^T \left( \frac{1}{3} \delta_{ij} - \epsilon_i^{T*} \epsilon_j^{T*} \right). \tag{18} \]
where we define \( I \equiv I_0 + I_{00}^{RD} \). The size of the load may be conveniently normalized by considering the ratio of the degree
two gravitational potential perturbations due to the direct effect of the load and the hydrostatic rotational bulge. Following the symbolism adopted in Willemann (1984), if we denote this ratio as $Q'$, then

$$
Q' = -\frac{G' \rho}{kT - kT + \alpha^2 \Omega^2 / (3\sqrt{5GM})},
$$

where the minus sign is included so that $Q'$ is positive for a mass excess. Replacing Eq. (19) in Eq. (18) yields

$$
I_{ij} = I \delta_{ij} + M \alpha Q'(m_* - m_o) \left( \frac{1}{3} \delta_{ij} - \varepsilon^L_i \varepsilon^L_j \right) - (m_* - m_o) \left( \frac{1}{3} \delta_{ij} - \varepsilon^Q_i \varepsilon^Q_j \right) - m \left( \frac{1}{3} \delta_{ij} - \varepsilon^Q_i \varepsilon^Q_j \right),
$$

(20)

where we use Eq. (10) and define the dimensionless coefficients

$$
\alpha \equiv \frac{1 + kL^2}{1 - kL^2 / kT^2},
$$

(21)

$$
m_* \equiv kL^2 \frac{\Omega^2 \alpha^2}{3GM},
$$

(22)

$$
m_o \equiv kL^2 \frac{\Omega^2 \alpha^2}{3GM},
$$

(23)

and

$$
m \equiv kL^2 \frac{\Omega^2 \alpha^2}{3GM}.
$$

(24)

3. TPW solutions without the effect of elastic energy

In this section we find TPW solutions for the case where we ignore the elastic energy in the lithosphere. We use two approaches. First, following previous work (Willemann, 1984; Matsuyama et al., 2006) we diagonalize the inertia tensor. Second, we minimize the total rotational energy of the system. Both approaches must, of course, yield the same result. In the next section we will compare these results to the solutions obtained by minimizing the total (rotational and elastic) energy of the planet.

3.1. Diagonalization of the inertia tensor

For a planet with no elastic lithosphere, TPW solutions can be found by diagonalizing the non-equilibrium inertia tensor since the equilibrium contributions are independent of the amount of reorientation. The validity of this approach will be confirmed in the next section by comparison with TPW solutions obtained using an energy minimization approach.

We use Eq. (20) to write the non-equilibrium inertia tensor (hence the super-script ‘NE’)

$$
I_{ij}^{NE} = M \alpha Q'(m_* - m_o) \left( \frac{1}{3} \delta_{ij} - \varepsilon^L_i \varepsilon^L_j \right) - (m_* - m_o) \left( \frac{1}{3} \delta_{ij} - \varepsilon^Q_i \varepsilon^Q_j \right),
$$

(25)

where we ignore the last term in the square brackets on the right-hand side of Eq. (20) since this term corresponds to the perturbation associated with the final rotation pole. Once again, the last term in Eq. (25) is associated with the remnant rotational bulge. We also ignore the spherically symmetric term, $I \delta_{ij}$, in Eq. (20) since this term will have no effect on the orientation and ordering of the principal axes, and thus it will not impact the ultimate reorientation of the pole.

We choose, with no loss of generality, a reference frame rotating with the planet, with the $z$-axis along the final rotation pole and the load center in the $x$-$z$ plane with a positive $x$-component, as shown in Fig. 1. We find the spherical coordinates $(\delta, \phi)$ of the paleopole which diagonalize the non-equilibrium inertia tensor in this frame. The amount of reorientation is given by the paleopole colatitude, $\delta$, and we denote the angular distance between the load center and the final rotation pole as $\theta^L_f$. The unit vectors associated with the final load center, the paleopole, and the final rotation pole are $\varepsilon^L_i = (\sin \theta^L_f, 0, \cos \theta^L_f)$, $\varepsilon^Q_i = (\sin \delta \cos \phi, \sin \delta \sin \phi, \cos \delta)$, and $\varepsilon^Q_j = (0, 0, 1)$, respectively. Replacing these unit vectors in the non-equilibrium inertia tensor [Eq. (25)] and setting the off-diagonal components to zero yields

$$
I_{13}^{NE} = 0 = \frac{Ma^2}{2} (m_* - m_o) \left[ -\alpha Q' \sin(2\theta^L_f) + \sin(2\delta) \cos \phi \right],
$$

$$
I_{23}^{NE} = 0 = \frac{Ma^2}{2} (m_* - m_o) \sin(2\delta) \sin \phi,
$$

Fig. 1. Reference frame rotating with the planet used in Sections 3.1, 3.2, and 4.2. The $z$-axis is aligned with the final rotation vector ($\Omega$) and the spherical coordinates for the load center and the paleopole are $(\theta^L_f, 0)$ and $(\delta, \phi)$, respectively.
\[
I_{12}^{\text{NE}} = 0 = \frac{Ma^2}{2} (m_* - m_0) \sin^2 \delta \sin(2\phi).
\] (26)

The last two equations imply \( \phi = 0^\circ \) or \( 180^\circ \), or \( \delta = 0^\circ \) or \( 180^\circ \). We can ignore the latter solution since \( I_{13}^{\text{NE}} \neq 0 \) in this case. The solutions for \( \phi = 0^\circ \) or \( 180^\circ \) constrain the paleopole location to the meridian connecting the load center with the final rotation pole. The inertia tensor diagonalization is completed by setting \( I_{11}^{\text{NE}} = 0 \). The TPW solutions are then given by \( \phi = 0^\circ \) and
\[
\delta = \frac{1}{2} \sin^{-1} \left[ Q' \alpha \sin(2\theta'_L) \right],
\] (27)

which agrees with Matsuyama et al. (2006). Note that choosing \( \phi = 180^\circ \) yields a reorientation angle of \( -\delta \), which corresponds to the same maximum principal axis rotational state. The smaller possible solutions for the rotation pole location since the inertia tensor perturbation is independent of the direction of rotation. That is, choosing the anti-pole of a given rotation pole yields the same maximum principal axis rotational state. The smaller TPW solution is physically more meaningful since it requires less work.

3.2. Rotational energy minimization

We provide an alternative solution for TPW in this section which illustrates that the alignment of the maximum principal axis of the inertia tensor with the rotation pole corresponds to the minimum rotational energy state for a given angular momentum. The rotational energy and angular momentum of the planet are given by

\[
K = \frac{1}{2} I_{ij} \Omega_i \Omega_j \quad \text{and} \quad L_i = I_{ij} \Omega_j,
\] (30)

where we adopt Einstein’s summation notation. We choose the reference frame rotating with the planet used in the previous section (Fig. 1). Substituting the rotation vector \( \Omega_i = (0, 0, \Omega) \) into Eq. (30), we can express energy and angular momentum as

\[
K = \frac{1}{2} I_{33} \Omega_3^2 \quad \text{and} \quad L_i = I_{33} \Omega_3 \delta,
\] (31)

respectively. The relevant inertia tensor components in the above equations are

\[
I_{13} = I + Ma^2 \left[ \frac{2}{3} m + (m_* - m_0) Q' \alpha \left( \frac{1}{3} - \cos^2 \theta'_L \right) \right. \\
- \left. (m_* - m_0) \left( \frac{1}{3} - \cos^2 \delta \right) \right],
\] (32)

where we have used Eq. (20) for the total inertia tensor with the unit vectors associated with the final load center and the paleopole (see last section).

We assume that angular momentum is conserved since the application of radially-directed loads anywhere on the surface cannot lead to torques. The magnitude of the angular momentum, \( L = (L_1^2 + L_2^2 + L_3^2)^{1/2} \), is a scalar and thus it is also conserved in a reference frame rotating with the planet. We can use this constraint to write the angular velocity as a function of the reorientation angle as

\[
\Omega(\delta, \phi) = \frac{L}{(I_{11}^{\text{NE}}(\delta, \phi) + I_{22}^{\text{NE}}(\delta, \phi) + I_{33}^{\text{NE}}(\delta, \phi))^{1/2}}.
\] (33)

This equation illustrates that angular momentum conservation leads to angular velocity variations as the inertia tensor changes and the planet reorients. We may use Euler’s equation to derive a simpler expression for the angular velocity. Angular momentum conservation in the rotating frame of the planet reads \( dL/dt + \mathbf{Q} \times \mathbf{L} = 0 \), where \( \mathbf{L} = (L_1, L_2, L_3) \) and \( \mathbf{Q} = (0, 0, \Omega) \) are the angular momentum vector and rotation vector respectively. The \( z \)-component of the angular momentum is conserved \( (dL_3/dt = 0) \) in our rotating reference frame since centrifugal forces cannot lead to torques around the \( z \)-axis. Hence, we can write the angular velocity as

\[
\Omega(\delta, \phi) = \frac{L_3}{I_{33}(\delta, \phi)}.
\] (34)

We set the constant \( L_3 \) as the \( z \)-component of the initial angular momentum by setting \( \Omega = \Omega^*, \theta'_L = \theta_L \), and \( \delta = 0 \) in Eq. (34):

\[
L_3 = \Omega^* I_{33} \left[ \theta'_L = \theta_L, \delta = 0 \right].
\] (35)

We rewrite the rotational energy to account for angular momentum conservation as

\[
K = \frac{L_3^2}{2 I_{33}(\delta, \phi)}.
\] (36)

Instead of finding the paleopole location which minimizes the rotational energy we can find the paleopole location which maximizes \( I_{33} \) (Appendix A). Setting \( \partial I_{33}/\partial \phi = 0 \) yields \( \sin \phi = 0 \), and thus, \( \phi = 0^\circ \) or \( 180^\circ \). That is, the rotation pole reorientation must occur along the great circle connecting the paleopole with the load center. Setting \( dL_3/d\delta = 0 \) yields

\[
\alpha Q' \sin(2\theta'_L) = \begin{cases} 
\sin(2\delta) & \text{for } \phi = 0^\circ, \\
-\sin(2\delta) & \text{for } \phi = 180^\circ.
\end{cases}
\] (37)

These two solutions correspond to the same rotation pole location, and they are equivalent to the solution found from the diagonalization of the non-equilibrium inertia tensor [Eq. (27)], as would be expected. Finally, we use the maximum condition \( d^2 I_{33}/d\delta^2 < 0 \) to find

\[
\cos(2\delta) > \alpha Q' \cos(2\theta'_L),
\] (38)

which is equivalent to the constraint \( I_{33}^{\text{NE}} > I_{11}^{\text{NE}} \), given by Eq. (28).
4. TPW solutions with the effect of elastic energy

Ojakangas and Stevenson (1989) have noted that the minimum total energy state for planets with purely elastic lithospheres may not correspond to maximum principal axis rotation because some of the energy is stored as elastic energy in the lithosphere as the rotation pole migrates. In this section we obtain TPW solutions by finding the minimum total energy state that includes the self-consistent elastic energy stored in the lithosphere.

4.1. Elastic energy in the lithosphere

Let us adopt a reference frame with the final rotation pole along the z-axis and the paleopole on the x-z plane with spherical coordinates $(\delta, 0)$, as shown in Fig. 2A. As we discussed above, the initial (before reorientation) planetary figure is given by the hydrostatic form (i.e., the form achieved by a rotating planet with no lithosphere). This form is given by

$$r_s(\theta, \phi) = a \left[ 1 - f_s(\cos^2 \gamma - 1/3) \right],$$

where $f_s$ is the initial flattening of the planet (defined as the difference between the equatorial and polar radii divided by $a$).

The angular distance between an arbitrary point with spherical coordinates $(\theta, \phi)$ on the surface of the planet and the paleopole, $\gamma$, can be found by using the spherical cosine law (see Fig. 2A):

$$\cos \gamma = \cos \delta \cos \theta + \sin \delta \sin \theta \cos \phi.$$  \hspace{1cm} (40)

The initial flattening can be written as

$$f_s = h_2^T \frac{a^3 \Omega^2}{2GM},$$  \hspace{1cm} (41)

where $h_2^T$ is the degree two $h$ fluid Love number for the planet with no lithosphere.

The planetary figure after reorientation of the rotation pole is given by

$$r = a \left[ 1 - f(\delta)(\cos^2 \gamma - 1/3) - f(\delta)(\cos^2 \theta - 1/3) \right],$$

where

$$f(\delta) = \frac{h_2^T a^3 \Omega^2}{2GM},$$  \hspace{1cm} (42)

and $h_2^T$ is the degree two $h$ fluid Love number for the planet with the elastic lithosphere. The angular velocity after reorientation of the rotation pole, $\Omega(\delta)$, is given by Eq. (34). The last two terms on the right-hand side of Eq. (42) represent the contribution to the planetary figure from the change in the centrifugal potential associated with a reorientation of the rotation pole and angular velocity variations. It is useful to consider some special cases. If the planet with lithosphere behaves with infinite rigidity (i.e., $h_2^T = 0$), then the final flattening is zero (i.e., $f(\delta) = 0$) and thus the final shape depends only on the initial flattening and the angular distance to the paleopole. Alternatively, if the planet remains hydrostatic ($h_2^T = h_2^T(\delta)$) throughout, then $f_s = f(\delta)$ and the final shape depends only on the angular distance to the final rotation pole, as would be expected. Finally, if the planet were to stop rotating ($\Omega(\delta) = 0$), then $f(\delta) = 0$ and the departure from a spherical form represents the permanent remnant bulge aligned with the initial rotation pole.

We represent the deformation due to reorientation as the difference between the two oblate shells associated with the initial and final rotational figures. The radial displacement of the lithosphere is given by

$$d_r = r - r_s = af(\delta)(\cos^2 \gamma - 1/3) - f(\delta)(\cos^2 \theta - 1/3).$$  \hspace{1cm} (45)

Our representation of the radial displacement is mathematically equivalent to that of Vening Meinesz (1947); however, our representation is physically more meaningful since it is explicitly associated with the difference between the initial and final planetary figures.

Vening Meinesz (1947) found equilibrium stress solutions for a thin, nearly spherical shell with a radial displacement of the form

$$d_r = af(\cos^2 \theta - 1/3).$$  \hspace{1cm} (46)

In this case, these stress solutions are

$$\sigma_{\theta \theta} = \frac{1}{5 + v} \frac{fE}{3} (3 \cos^2 \theta + 1),$$  \hspace{1cm} (47)

$$\sigma_{\phi \phi} = \frac{1}{5 + v} \frac{fE}{3} (9 \cos^2 \theta - 5),$$  \hspace{1cm} (48)

where $E$ is Young’s modulus, $v$ is Poisson’s ratio, and tensile stresses are positive.

The stress solutions for the second term on the right-hand side of Eq. (45) are given by Eqs. (47) and (48) with a negative sign. The same solutions are valid for the stresses associated with the first term on the right-hand side of Eq. (45), but in a
reference frame with the z-axis along the paleopole since γ is the angular distance to the pole if we denote the stresses in a reference frame reoriented such that the z-axis is along the poleopole as \( \sigma_{ij}^{*} \), then

\[
\sigma_{\theta \theta}^{*} = \left( \frac{1}{5 + \nu} \right) f_0 E \left( \frac{2}{3} (3 \cos^2 \gamma + 1) \right), \quad (49)
\]

\[
\sigma_{\phi \phi}^{*} = \left( \frac{1}{5 + \nu} \right) f_0 E \left( \frac{2}{3} (9 \cos^2 \gamma - 5) \right). \quad (50)
\]

Following the analysis of Melosh (1980), we account for the reorientation of the unit vectors \( e_{\theta}', e_{\phi}' \) (of a reference frame with the z-axis along the initial rotation vector) relative to the unit vectors \( e_{\theta}, e_{\phi} \) (of a reference frame with the z-axis along the final rotation pole), as illustrated in Fig. 2. In a reference frame with the final rotation pole along the \( z \)-axis, the stress solutions for the first term in Eq. (45), which are associated with the poleopole, are given by

\[
\sigma_{\theta \theta} = \sigma_{\theta \theta}^{*} \cos^2 \psi + \sigma_{\phi \phi}^{*} \sin^2 \psi = \frac{E f_0}{3(5 + \nu)} \left( 6 \sin^2 \gamma \cos^2 \psi + 9 \cos^2 \gamma - 5 \right),
\]

\[
\sigma_{\phi \phi} = \sigma_{\theta \theta}^{*} \sin^2 \psi + \sigma_{\phi \phi}^{*} \cos^2 \psi = \frac{E f_0}{3(5 + \nu)} \left( -6 \sin^2 \gamma \cos^2 \psi + 3 \cos^2 \gamma + 1 \right),
\]

\[
\sigma_{\theta \phi} = -\left( \sigma_{\theta \theta}^{*} - \sigma_{\phi \phi}^{*} \right) \sin \psi \cos \psi = -\frac{E f_0}{5 + \nu} \sin^2 \gamma \sin(2\psi), \quad (51)
\]

where we use Eqs. (49) and (50) and \( \psi \) is the reorientation angle for the unit vectors (Fig. 2).

We can find expressions for the terms which depend on the reorientation angle between the unit vectors in Eq. (51) using spherical trigonometry. Combining the spherical cosine law, \( \cos \delta = \cos \gamma \cos \theta + \sin \gamma \sin \theta \cos \psi \) (see Fig. 2), with Eq. (40) yields

\[
\sin \gamma \cos \psi = \cos \delta \sin \theta - \sin \delta \cos \theta \cos \phi. \quad (52)
\]

Similarly, we can use the spherical sine law to write

\[
\sin \gamma \sin \psi = \sin \delta \sin \phi. \quad (53)
\]

Finally, the equilibrium stress solutions which include all the terms in Eq. (45) are given by \( \sigma_{ij} \equiv \sigma_{ij}^{*} - \sigma_{ij}' \). Using Eqs. (47), (48), and (51) yields

\[
\sigma_{\theta \theta} = \frac{E}{3(5 + \nu)} \left[ f_0 (6 \sin^2 \gamma \cos^2 \psi + 9 \cos^2 \gamma - 5) - f(\delta) (3 \cos^2 \theta + 1) \right],
\]

\[
\sigma_{\phi \phi} = \frac{E}{3(5 + \nu)} \left[ f_0 (-6 \sin^2 \gamma \cos^2 \psi + 3 \cos^2 \gamma + 1) - f(\delta) (9 \cos^2 \gamma - 5) \right],
\]

\[
\sigma_{\theta \phi} = -\frac{E f_0}{5 + \nu} \sin^2 \gamma \sin(2\psi). \quad (54)
\]

Reorientation and angular velocity variations may produce global tectonic patterns on the lithosphere if the stresses exceed the yield strength. The nature of the expected faulting is defined by the magnitude and orientation of the principal stresses relative to the stress-free planet surface, following Anderson (1951), Vening Meinesz (1947) and Melosh (1980) found solutions for the stresses associated with reorientation of the rotation pole, and Melosh (1977) found solutions for the stress of a despun planet. We extend these studies with Eqs. (54) by providing solutions for the stresses associated with the simultaneous reorientation of the rotation pole and despinning of the planet. Using Eq. (54), we can recover Eqs. (1)–(3) of Melosh (1980) if we assume that the rotation rate remains constant and set \( f_0 = f(\delta) \), while we can recover Eq. (23) of Melosh (1977) if we assume that the rotation pole remains fixed and set \( \delta = \psi = 0 \) and \( \gamma = \theta \) (see Fig. 2).

We use a thin shell approximation to find the total elastic energy of the lithosphere, \( U \), by integrating the elastic energy density across the volume of the lithospheric shell:

\[
U = \frac{a^2}{2} T_e \int_S \epsilon_{ij} \sigma_{ij} \ dS = \frac{1}{2} \int_S \int_0^{2\pi} d\phi \sin \theta \left( \sigma_{\theta \theta} \epsilon_{\theta \theta} + \sigma_{\phi \phi} \epsilon_{\phi \phi} + 2 \sigma_{\theta \phi} \epsilon_{\theta \phi} \right), \quad (55)
\]

where \( S \) denotes the complete solid angle and \( T_e \) is the thickness of the elastic lithosphere (Landau and Lifshitz, 1969, p. 54). Applying the stress–strain relations:

\[
\epsilon_{\theta \theta} = \frac{\sigma_{\theta \theta} - \nu \sigma_{\phi \phi}}{E}, \\
\epsilon_{\phi \phi} = \frac{\sigma_{\phi \phi} - \nu \sigma_{\theta \theta}}{E}, \\
\epsilon_{\theta \phi} = \frac{\sigma_{\theta \phi}}{2\mu},
\]

in Eq. (56) gives

\[
U = \frac{a^2}{2} T_e \int_0^{2\pi} \int_0^\pi d\phi \sin \theta \left[ \frac{1}{E} (\sigma_{\theta \theta}^2 + \sigma_{\phi \phi}^2)
- 2\nu \sigma_{\theta \theta} \sigma_{\phi \phi}
+ \frac{\sigma_{\phi \phi}^2}{\mu} \right]. \quad (57)
\]

Using Eqs. (40) and (52)–(54) to evaluate the integral in Eq. (57) yields

\[
U = \frac{a^2 E T_e}{(5 + \nu)} \left[ 16 \pi - 2 \frac{f(\delta)^2 + 2 f_0^2 - f(\delta) f_0}{45} - 3 f(\delta) f_0 \cos(2\delta) \right]. \quad (58)
\]

We can recover Eq. (42) of Vening Meinesz (1947) if we ignore rotation rate variations and assume \( f(\delta) = f_0 \) in Eq. (58).

### 4.2. Total energy minimization

The total energy of a planet after reorientation is the sum of the rotational kinetic energy and the elastic energy in the lithosphere due to departures from its equilibrium shape. Using
Eqs. (36) and (58), the total energy can be written as
\[
H(\delta) = U + K = \frac{L^2}{2I_3} + \kappa \left[2f(\delta)^2 + 2f_0^2 - f(\delta)f_0 - 3f(\delta)f_0 \cos(2\delta)\right],
\]
where we define
\[
\kappa \equiv \frac{a^2 ET_e 16\pi}{(5 + v) 45},
\]
and \(I_{33}(\delta), L_3\), and \(f(\delta)\) are given by Eqs. (32), (35), and (44), respectively.

Let us adopt the reference frame used in Sections 3.1 and 3.2 (Fig. 1), with the final rotation vector at the north pole and the load center on the \(x-z\) plane with spherical coordinates \((\theta_f^l, \phi)\). We find the paleopole orientation which minimizes the total energy by varying its spherical coordinates \((\delta, \phi)\). We may assume that reorientation occurs along the great circle connecting the load center and the paleopole \((\phi = 0)\) since we found that this corresponds to the minimum energy state for the rotational energy alone in Section 3.2 and the elastic energy is independent of \(\phi\) (the rotation pole reorientation angle is given by \(\delta\) in our reference frame). Using Eq. (59) and setting \(dH/d\delta = 0\) yields (see Appendix B)
\[
\sin(2\delta) = \frac{\alpha}{\beta} Q \sin\left(2\theta_f^l\right),
\]
where
\[
\beta \equiv 1 + \frac{4k_f f_0 (3I_3^2 \Omega^2 + 4L_3^2 Ma^2 m_0)}{L_3^2 Ma^2 (m_* - m_0)(I_{33} \Omega^2 + 4k_f f_0 [4f(\delta) - f_0] - 3f_0 \cos(2\delta))}.
\]

Comparison of Eqs. (27) and (61) indicates that the effective load size is reduced by factor of \(\beta\) when the elastic strain energy of the lithosphere is included. It is useful to consider some limiting cases of Eqs. (61) and (62). The first important case is that of a planet with no elastic energy in the lithosphere. In this case, \(\kappa = 0\) and we recover Eq. (27). We can also recover Eq. (27) if we consider the case where \(h^L = 0\) and therefore \(f = f_0 = 0\) and \(\beta = 1\), since there would be no radial displacement of the lithosphere in this case.

For a hydrostatic planet (i.e., for a planet with no elastic lithosphere), the surface load is fully compensated \((k^2_L = -1)\), there is no remnant rotational bulge \((k^T = k^T_* = 0)\) and \(m_0 = m_*,\) and there is no elastic energy \((\kappa = 0)\); thus the factor \(\alpha/\beta\) [see Eqs. (21) and (62)] in Eq. (61) becomes singular. This singularity arises because the rotation pole of a hydrostatic planet becomes unstable. From a minimum energy point of view, the rotational energy is the same for any rotation pole orientation, and in the inertia tensor diagonalization approach, the maximum principal axis orientation is singular since the non-hydrostatic inertia tensor has no preferred orientation. The TPW solution obtained by neglecting elastic energy [Eq. (27)] exhibits the same singularity in the hydrostatic case.

We can find a useful approximation to the TPW solution if we make some simplifying assumptions. Since perturbations to the angular velocity and inertia tensor are small (relative to the unperturbed initial state) we can assume \(f \sim f_0, I_{33} \sim I_0,\) and \(L_3 \sim I_0 \Omega^2\) to simplify Eq. (62). If we further assume that \(m_0 Ma^2 I_0 \ll 1\) and \(f_0^2 / (I_0 \Omega^2) \ll 1,\) which are good approximations for nearly spherical \((m_0, f_0 \ll 1)\) planetary objects, then Eq. (62) becomes
\[
\beta \approx 1 + \frac{16\pi}{5} \left(\frac{a^3}{GM^2}\right) \left(\frac{ET_e}{5 + v}\right) \left[\frac{h^T_1}{k^T_* - k^T_2}\right].
\]
This approximation greatly simplifies Eq. (62) since \(\beta\) becomes independent of the load size and location, and of the reorientation angle. We will discuss the accuracy of this approximation in the next section by comparing it with the solution given by Eq. (62).

An additional constraint on the reorientation angle can be found by considering the differential energy change as the planet undergoes reorientation. The differential energy change cannot be positive at any stage during reorientation to avoid energy gain: i.e., \(dH/d\delta \leq 0\). Since the energy minimum condition is given by \(dH/d\delta = 0\), the differential energy constraint implies that the total energy must decrease monotonically to the minimum energy value during reorientation. If we use the same approximations listed above we can write the differential energy constraint as (Appendix B)
\[
\frac{1}{Ma^2 \Omega^2} \frac{dH}{d\delta} = \frac{1}{2} (m_* - m_0) \left[\sin(2\delta) - \frac{\alpha}{\beta} Q \sin\left(2\theta^l_f\right)\right] \leq 0.
\]

5. Application to Mars

The equatorial location of the Tharsis rise on Mars is commonly explained by a reorientation of the planet due to TPW. Melosh (1980), for example, suggested that Mars could have been reoriented by as much as 25° by this loading event. His principal axis analysis was extended by Willemann (1984) to include inertia tensor contributions associated with a remnant rotational bulge. Willemann (1984) concluded that the amount of reorientation was less than 18° or greater than 72° if Tharsis represents a load with upper size bound \(Q = 1.74\), though he excluded the latter (large-TPW) solution on the basis of probability arguments. Assumptions made in the Willemann (1984) study led to the conclusion that the predicted equilibrium pole position is independent of the thickness of the elastic lithosphere. Matsuyama et al. (2006) revisited this problem using a Love number formulation and their results identified a sensitivity to lithospheric thickness. [Their theory differs from the earlier analysis by the appearance of the term \(\alpha\) in Eq. (27); setting \(\alpha = 1\) in this equation yields the expressions derived by Willemann (1984).] Furthermore, they argued that diagonalization of the non-equilibrium inertia tensor permits large excursions (≥65°) of the martian rotation pole, even when the stabilizing effect of the remnant rotational bulge is included.

None of these previous studies considered the additional stabilizing effect of elastic energy stored in the lithosphere. In this section we present the results of applying the TPW solutions of Section 4, which include this physics, to an investigation of the rotational stability of Mars.

The support of tidal and surface loads for a small planet like Mars is primarily due to membrane stresses for low de-
Reorientation of planets with lithospheres

Table 1
Effects of lithospheric thickness \((T_e)\) upon the Mars model parameters

<table>
<thead>
<tr>
<th>(T_e) (km)</th>
<th>(k_f^L)</th>
<th>(k_f^T)</th>
<th>(h_s^L)</th>
<th>(T_e)</th>
<th>(T_e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>−0.9164</td>
<td>1.109</td>
<td>2.025</td>
<td>1.2708</td>
<td>1.805</td>
</tr>
<tr>
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<td>−0.8529</td>
<td>1.037</td>
<td>1.890</td>
<td>1.1686</td>
<td>1.731</td>
</tr>
<tr>
<td>200</td>
<td>−0.6906</td>
<td>0.8475</td>
<td>1.538</td>
<td>1.0822</td>
<td>1.426</td>
</tr>
<tr>
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<td>−0.5753</td>
<td>0.7124</td>
<td>1.288</td>
<td>1.0625</td>
<td>1.320</td>
</tr>
<tr>
<td>400</td>
<td>−0.4877</td>
<td>0.6097</td>
<td>1.097</td>
<td>1.0537</td>
<td>1.255</td>
</tr>
</tbody>
</table>

Note: We adopt the 5-layer model of martian structure described by Bills and James (1999). The Love numbers for \(T_e = 0\) are \(k_f^T = 1.186\) and \(h_s^L = 2.186\). The factors of \(\alpha\) and \(\beta\) are given by Eqs. (21) and (63), respectively.

degree harmonics, including degree two (Turcotte et al., 1981). On Mars, the hemispheric dichotomy (degree one) and Tharsis (degree two) have survived for 4 Gyr, and viscoelastic relaxation models indicate that long wavelength loads could be maintained over geologic time scales (Zhong and Zuber, 2000; Zuber et al., 2000). In application to the TPW of Mars driven by the rise of Tharsis, the relevant elastic lithospheric thickness is that at the time the load was emplaced. Elastic thicknesses at the time of loading have been estimated using gravity and topography data from the Mars Global Surveyor (MGS) spacecraft: McGovern et al. (2004) obtained values of elastic lithospheric thicknesses \(\sim 50\) km for the Tharsis Montes, and Turcotte et al. (2002) found a global average value \(\sim 100\) km, which we adopt as the fiducial value.

We adopt the 5-layer model of martian structure described by Bills and James (1999). The elastic parameters of the lithosphere are \(\mu = 45\) GPa and \(\nu = 0.25\). The fluid Love number values, together with the factor \(\alpha\) [Eq. (21)] and the approximate expression for the factor \(\beta\) [Eq. (63)] are provided on Table 1 for a sequence of elastic lithospheric thicknesses ranging from 50 to 400 km.

To quantify the stabilizing effect of the elastic strain energy stored in the lithosphere, Fig. 3 compares the TPW angle as a function of the final load colatitude for solutions without [Eq. (27)] and with [Eqs. (61)] this physics included. The figure also shows solutions generated using the approximation given by Eq. (63). Each frame shows solutions for \(T_e = 100\) km and different load sizes represented by the parameter \(Q'\) (as labeled). Comparison of Figs. 3B and 3C demonstrates that the approximation specified in Eq. (63) is valid over a broad range of load sizes, load locations, and TPW angles.

The effective size of the TPW driving load forcing is effectively reduced due to the stabilizing effect of elastic energy stored in the lithosphere. For \(T_e = 100\) km, the load is reduced by a factor of \(\beta \sim 1.7\) (see Table 1). Therefore, the TPW solution for \(Q' = 1.7\) with the elastic strain energy included (Figs. 3B and 3C) and the solution for \(Q' = 1\) without the elastic energy included (Fig. 3A) are nearly identical.

The stabilizing effect of the elastic energy decreases (i.e., \(\beta\) decreases) with increasing thickness of the lithosphere, as shown in Table 1. This result is somewhat surprising since a planet with a thicker lithosphere is able to store more elastic energy. However, Eq. (63) shows that a planet with a thicker lithosphere also experiences less deformation, which leads to less elastic energy in the lithosphere. Moreover, the larger remnant rotational bulge of a planet with a thicker lithosphere stabilizes the rotation pole, which leads to smaller strains and elastic energy. Mathematically, \(\beta\) decreases because the factor \((k_f^T - k_f^L)/h_s^L^2\) increases faster than \(T_e\) as \(T_e\) increases.

We can find specific TPW solutions for Mars by assuming that Tharsis represents an axisymmetric load currently located at 83° colatitude (Zuber and Smith, 1997) and with size \(Q' = 1.74\) (Willemann, 1984). Fig. 4 shows the TPW angle as a function of the final load colatitude for the case \(T_e = 100\) km. Fixing the Tharsis colatitude to 83° provides an additional constraint shown by the dotted line in Fig. 4. There are two possible solutions. If Tharsis formed near the equator, the addition of the elastic energy term reduces the predicted TPW angle from 15° to 8°. On the other hand, if Tharsis formed near the rotation pole, the TPW angle increases from 75° to 82° when the elastic energy term is included. Fig. 5 shows that the differential energy change for the small (8°) and large (82°) TPW solutions...
remains negative at any stage of the reorientation, as required by energy considerations.

The large TPW solution is worthy of further comment since the TPW angle is predicted to increase when elastic energy in the lithosphere is included in the theory. For this solution, the initial load center is almost aligned with the paleopole since the final Tharsis colatitude is 83° and the TPW angle is 82°. In this case, the compensated load acts to ‘cancel’ the remnant rotational bulge (Matsuyama et al., 2006). As noted above, the effect of the elastic energy is to decrease the effective load size; therefore, in order to reorient the pole to its present location, the initial Tharsis center must be closer to the paleopole to compensate for this size reduction. The larger reorientation of the rotation pole is thus a result of reducing the effective load size of Tharsis (by including the elastic energy) and fixing the final angular distance between the rotation pole and Tharsis. Note, in this regard, that assuming a Tharsis load size smaller than \( Q' = 1.74 \) would also increase the amount of predicted reorientation. Finally, we also note that the near cancellation of the remnant rotational bulge by Tharsis in the large-TPW solution would leave the planet in a relatively unstable rotational state; that is, a subsequent episode of TPW may have been triggered (along the great circle perpendicular to Tharsis) by a small perturbation to the inertia tensor (Matsuyama et al., 2006).

6. Discussion

We have demonstrated that principal axis rotation is not the minimum energy state for a planet with an elastic lithosphere that has been reoriented due to TPW (see also Ojakangas and Stevenson, 1989). As the planet undergoes reorientation toward principal axis rotation, which corresponds to the minimum rotational energy state, elastic energy is stored within the deforming lithosphere. Thus, the minimum total energy state is achieved before the planet reaches principal axis rotation. We derived a new TPW solution by minimizing the total energy of the planet [Eqs. (61)–(63)]. We furthermore showed that including elastic energy within the lithosphere leads to a reduction in the effective size of the surface mass load driving the TPW. This reduction becomes more pronounced for planets with progressively thinner lithospheres (over the range of values considered in Table 1), since these are subject to progressively larger strains.

As an illustration of the new theory, we applied it to revisit the rotational stability of Mars in response to loading by the Tharsis volcanic province. There are two possible TPW scenarios given the present colatitude of Tharsis and its estimated (upper bound) size (\( Q' = 1.74 \)): for the case of an elastic lithosphere of thickness \( T_e = 100 \) km, Tharsis formed near either the martian equator and little TPW (<8°) occurred, or near the martian paleopole and migrated to the present equator via a large TPW event (>82°). For comparison, the TPW solutions derived without including the effect of stored elastic energy predicted more permissive ranges of reorientation; smaller than 15° or larger than 75° for the same load size for Tharsis and...
adopted lithospheric thickness. If Tharsis formed near the paleopole, the remnant rotational bulge is effectively canceled by the loading, thereby enabling a subsequent, large excursion of the rotation pole.

Finally, we note that in studies of the Earth’s long-term rotational stability (e.g., Ricard et al., 1993; Steinberger and O’Connell, 1997; Richards et al., 1999; Greff-Lefftz, 2004) it has been assumed that the rotational bulge will ultimately (i.e., in the secular limit of the governing equations) adjust to any new orientation of the rotation axis. That is, it has been assumed, explicitly or implicitly, that there is no stabilization by a remnant rotational bulge. Spada et al. (1996) considered the long-term rotational stability of Venus, Earth, and Mars under the same assumption. However, the development of an elastic lithosphere implies that a nonzero remnant rotational bulge may contribute to the rotational stability. We adopt planetary models with a single, uniform elastic plate. Therefore, our theory overestimates the rotational stability of planets like Earth with a broken lithosphere since the remnant rotational bulge is expected to be smaller (but nonzero) in this case.

Acknowledgments

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Appendix A. Rotational energy minimization

In a reference frame rotating with the planet, with the $z$-axis along the final rotation pole and the load center in the $x-z$ plane with a positive $x$-component (Fig. 1), the angular distance between the load center and the paleopole can be found using the spherical cosine law:

$$\cos \theta_L = \cos \theta_f \cos \delta + \sin \theta_f \sin \delta \cos \phi.$$  \hspace{1cm} (A.1)

Taking the derivative of Eq. (A.1) with respect to $\phi$ yields

$$\frac{\partial \theta_f}{\partial \phi} = \frac{\sin \theta_f \sin \delta \sin \phi}{\cos \theta_f \sin \delta \cos \phi - \sin \theta_f \cos \delta}.$$  \hspace{1cm} (A.2)

We find the extreme conditions for $I_{33}$ by taking the derivative of Eq. (32) with respect to $\phi$ and $\delta$. Setting $\partial I_{33}/\partial \phi = 0$ yields

$$0 = \cos \theta_L \sin \theta_f \left( \frac{\sin \theta_f \sin \delta \sin \phi}{\cos \theta_f \sin \delta \cos \phi - \sin \theta_f \cos \delta} \right),$$  \hspace{1cm} (A.3)

where we use Eq. (A.2). For an arbitrary final load colatitude, $\theta_f$, Eq. (A.3) implies $\phi = 0^\circ$ or $180^\circ$, or $\phi = 0^\circ$ or $180^\circ$. We can eliminate the solutions with $\delta = 0^\circ$ or $180^\circ$ since these solutions are independent of the location of the load center and are thus unphysical. Using Eq. (32) with $\phi = 0^\circ$ or $180^\circ$ and setting $dI_{33}/d\delta = 0$ yields

$$\frac{1}{Ma^2} \frac{dI_{33}}{d\delta} = \frac{2m_0 L_3^2}{3 d\delta} \pm \frac{(m_a - m_0) \alpha Q' \sin(2\theta_L^f)}{m_a - m_0} \sin(2\delta) = 0,$$  \hspace{1cm} (A.4)

where the plus and minus signs correspond to $\phi = 0^\circ$ and $180^\circ$, respectively, and we use $d\theta_f/\delta = \pm 1$ (Fig. 6). The first term on the right-hand side of Eq. (A.4) vanishes for $dI_{33}/d\delta = 0$ since

$$\frac{dm}{d\delta} = \frac{2m_0 L_3^2 dI_{33}}{\Omega^2 I_{33}^2} d\delta,$$ \hspace{1cm} (A.5)

where we use Eqs. (23), (24), and (34). The solutions for Eq. (A.4) are

$$\alpha Q' \sin(2\theta_L^f) = \begin{cases} \sin(2\delta) & \text{for } \phi = 0^\circ, \\ -\sin(2\delta) & \text{for } \phi = 180^\circ. \end{cases}$$  \hspace{1cm} (A.6)

These two solutions correspond to the same rotation pole location. Using Eq. (32) with $\phi = 0^\circ$ to calculate the maximum condition $d^2 I_{33}/d\delta^2 < 0$ gives

$$\alpha Q' \cos(2\theta_L^f) - \cos(2\delta) < \frac{2m_0 L_3^2}{3(m_a - m_0) \Omega^2 I_{33}^2} \frac{d^2 I_{33}}{d\delta^2} < 0,$$  \hspace{1cm} (A.7)

where we use Eqs. (A.4) and (A.5), and $d\theta_f/\delta = 1$ since we assume $\phi = 0^\circ$ (Figs. 6A and 6B).

Appendix B. Total energy minimization

The derivative of Eq. (59) with respect to $\delta$ with $\phi$ set to zero is

$$\frac{dH}{d\delta} = -\frac{L_3^2}{2I_{33}^2} \frac{dI_{33}}{d\delta} + 6\kappa f_0 \sin(2\delta) + \kappa \frac{df}{d\delta} \left[ 4f - f_0 - 3f_0 \cos(2\delta) \right].$$  \hspace{1cm} (B.1)

Using Eqs. (34), (43), and (44) yields

$$f(\delta) = \frac{f_0 L_3^2}{\Omega^2 I_{33}^2} \frac{df}{d\delta}$$ \hspace{1cm} (B.2)

and therefore

$$\frac{df}{d\delta} = -\frac{2f_0 L_3^2}{\Omega^2 I_{33}^2} \frac{dI_{33}}{d\delta}.$$  \hspace{1cm} (B.3)
Replacing Eq. (B.3) in Eq. (B.1) gives
\[
\frac{dH}{d\delta} = -\frac{L_3^2}{2I_{33}} \frac{dI_{33}}{d\delta} \left[ 1 + \frac{4\kappa f_0}{\Omega^2 I_{33}} \left( 4f - f_0 - 3f_0 \cos(2\delta) \right) \right] + 6\kappa f_0 \sin(2\delta). \tag{B.4}
\]

Using Eqs. (A.4) and (A.5) yields
\[
\frac{dI_{33}}{d\delta} = -2I_3 \left( 1 + \frac{4M a^2 m_0 L_3^2}{3\Omega^2 I_{33}^3} \right)^{-1} \left[ 1 + \frac{4\kappa f_0}{\Omega^2 I_{33}} \left( 4f - f_0 - 3f_0 \cos(2\delta) \right) \right] + 6\kappa f_0 \sin(2\delta). \tag{B.5}
\]

where \( I_{33} \) is given by Eq. (32) with \( \phi = 0 \). We replace Eq. (B.5) in Eq. (B.4) to obtain
\[
\frac{dH}{d\delta} = \frac{L_3^2}{I_{33}} \frac{dI_{33}}{d\delta} \left[ 1 + \frac{4M a^2 m_0 L_3^2}{3\Omega^2 I_{33}^3} \right]^{-1} \left[ 1 + \frac{4\kappa f_0}{\Omega^2 I_{33}} \left( 4f - f_0 - 3f_0 \cos(2\delta) \right) \right] + 6\kappa f_0 \sin(2\delta). \tag{B.6}
\]

Finally, using Eq. (32) with \( \phi = 0 \) to replace \( I_{13} \) in Eq. (B.6) yields
\[
\frac{1}{Ma^2\Omega^2} \frac{dH}{d\delta} = \frac{1}{2} \gamma(m_1 - m_0) \left[ \frac{\beta \sin(2\delta) - \alpha Q \sin(2\delta)}{\Omega^2} \right], \tag{B.7}
\]

where \( \beta \) is given by Eq. (62) and
\[
\gamma = \frac{3L_3^2}{\Omega^2} \left[ I_{33} \Omega^2 + 4\kappa f_0 \left( 4f - f_0 - 3f_0 \cos(2\delta) \right) \right] \left( 3I_{33}^2 \Omega^2 + 4L_3^2 Ma^2 m_0 \right). \tag{B.8}
\]

If we assume \( f \sim f_0, L_3 \sim I_0 \Omega^2, \kappa f_0^2/(I_0 \Omega^2) \ll 1, \) and \( m_0 Ma^2/I_0 \ll 1, \) then \( \gamma \sim 1. \)

**References**


