1. Here we’ll examine the thermal skin-depth problem, where a periodic variation in surface temperature propagates downwards.

In the simple Cartesian heat-conduction equation \( \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \) we can use separation of variables. We’ll write \( T = A(t)B(z) \) and assume that the variation in \( T \) is sinusoidal: \( A = \exp(i\omega t) \), where \( \omega \) is the angular frequency.

**a)** Show that in the case that the surface temperature is given by \( \Delta T \exp(i\omega t) \) and the temperature perturbations go to zero as \( z \) goes to infinity, the solution is given by:

\[
Re(T) = \Delta T \cos \left( \omega t - \sqrt{\frac{\omega}{2\kappa}} z \right) e^{-\sqrt{\frac{\kappa \omega}{2\kappa}} z}
\]

Hint: \( \sqrt{i} = \pm(1 + i) \); see also Turcotte and Schubert, *Geodynamics*, if you get stuck.

This shows that the thermal “skin depth” is given by \( z \sim (\kappa / \omega)^{1/2} \) as we’d expect from \( t \sim d^2 / \kappa \), and that there is a phase lag in the temperature response which varies with depth.

**b)** Write a numerical (finite-difference) code to solve the skin-depth problem for two cases: 1) constant thermal diffusivity; and 2) temperature-dependent diffusivity which varies as \( \kappa_0 (T_0 / T)^2 \) where \( \kappa_0 \) is the diffusivity at the mean temperature \( T_0 \).

Show that your code can reproduce the two following cases, where \( T_0 = 300 \) K, \( \Delta T = 20 \) K, \( \omega = 1e^{-6} \) rad s\(^{-1} \), \( \kappa_0 = 1e^{-6} \) m\(^2\) s\(^{-1} \):

![Figure 1](image)

Figure 1. Panel a) shows comparison between numerical and analytical results, and b) shows the results for constant and variable thermal diffusivity. These curves are plotted at \( \cos(\omega t) = 0 \) after any initial transients have damped.
2. Now we’ll think about the evolving temperature distribution inside a conductive sphere.

a) Write a finite-difference code to track the temperature evolution in a uniform sphere. Compare with the analytical solution of Carslaw & Jaeger for a sphere initially at a temperature $T_0$ where the surface temperature is increased to $T_1$ at time $t=0$:

$$T(r, t) = T_1 + \frac{2R(T_1 - T_0)}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi r}{R} \exp \left( -\frac{\kappa n^2 \pi^2}{R^2} t \right)$$

where $R$ is the radius, and $\kappa$ is the diffusivity. Reproduce the results in Figure 2a below, which plots the analytical and numerical solutions for $T_0=0$, $T_1=100$ K, $R=100$ km, $\kappa=1.\times10^{-6}$ m$^2$/s at $t=10.65$ Myr.

b) Now add in heat production (at a constant rate) to reproduce Figure 2b. This plots the temperature structure at $t=10.65$, 31.96 and 106.3 Myr with heat production at a rate of $10^{-10}$ W/kg in the bottom half of the sphere only. The specific heat capacity is $10^3$ J/kg K and the initial temperature is zero everywhere. The surface temperature is zero throughout.

[Optional:] c) The time constant for a sphere is $\sim R^2/\pi^2 \kappa$ or about 32 Myr. So by 106.3 Myr the temperature should be approaching steady state. Show that for the case in Fig 2b the steady-state solution is given by

$$T = T_0 + \frac{HR}{12\kappa C_p} (R^2 - 2r^2) \text{ for } r< R/2$$

and

$$T = T_0 + \frac{HR^3}{24\kappa C_p} \left( \frac{1}{r} - \frac{1}{R} \right) \text{ for } r> R/2$$

Hint: You’ll need two matching conditions at the interface ($r=R/2$). You can also check that the solution is correct by comparing the total rate of heat production with the total rate of heat loss at the surface.