0. Here I want you to write your own “dumb” fourier transform just so you get a feeling for how it works.

a) Download the short (20 point) profile \( h_k \) from
http://www.es.ucsc.edu/~fnimmo/eart290c_17/week2/shortprofile.txt

b) This profile is so short that you don’t need an FFT, you can just do it in the obvious brute-force manner, by looping over \( k \) and \( n \) to calculate all the \( H_n \)'s:

\[
H_n = \sum_{k=0}^{N-1} h_k e^{-2\pi i kn/N}
\]

Note that the \( H_n \)'s are complex even though the input data is real.

What the two dominant wavelengths of the signal, and what are their respective amplitudes (\( |H_n| \))?

c) If you write out the \( H_n \) array, you will see that they occur in complex conjugate pairs, with the symmetry axis being about the mid-point of the array. You can think of the first half of the array as representing positive wavenumbers, and the second half representing negative wavenumbers. [Recall that the max. and min. absolute wavenumbers are \( \pi/\Delta x \) and \( 2\pi/\Delta x \), respectively, so that to have \( N \) points in total you need to include both positive and negative wavenumbers]

A plot of \( H_n \) vs. wavenumber should look like this:

![Plot of H_n vs. wavenumber](image)

\[ -2.51 \quad -1.26 \quad 0 \quad 1.26 \quad 2.51 \]
\[ 0 \quad 10 \quad 20 \quad 30 \]

\( H_n \)

Wavenumber

d) Calculate \( \sum_{k=0}^{N-1} |h_k|^2 \) and \( \frac{1}{N} \sum_{n=0}^{N-1} |H_n|^2 \) and verify that Parseval's theorem is satisfied.

e) Also verify that by calculating \( h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{2\pi i kn/N} \) you can recover the original profile.
1. Here we’ll look at a simple example of spectral leakage.
   a) Download three files, the first of which is
      
      http://www.es.ucsc.edu/~fnimmo/eart290c_17/week2/profile3l.dat

      The other two have the same address and are profile3i.dat and profile3s.dat. The format for each is distance, value, and then two dummy entries. The units don’t really matter in this case.

      These are three samples of different lengths from the same synthetic profile, with l, i and s standing for long, intermediate and short.

   b) Plot the power spectra of all three profiles, as log(power) vs. log(wavelength).

   c) The two shorter samples can be thought of as the long profile multiplied by a boxcar filter in the space domain. What is the effect of such a boxcar filter in the frequency domain? What effect does taking shorter samples have on the power spectrum that you obtain?

2. Here we’ll examine some simple applications of fourier transforms.
   a) Download the data set
      
      http://www.es.ucsc.edu/~fnimmo/eart290c_17/week2/p3.cli

      This is an evenly-spaced topographic profile, format: distance (km), elevation (m).

   b) Plot the original data and a low-pass filtered version. The low-pass filter should have a value of 0.5 at 20 km wavelength and should vary smoothly from 0 at 15 km to 1 at 25 km (e.g. use a cosine taper).

   c) Flexure. In the frequency domain, the deflection of an elastic crust \( W_n \) due to a topographic load of amplitude \( H_n \) is given by

   \[
   W_n = H_n \left( \frac{D k_n^4}{g \rho} + \frac{\Delta \rho}{\rho} \right)^{-1}
   \]

   where \( \rho \) is the load density, \( \Delta \rho \) is the density contrast between the load and the material beneath the crust, \( g \) is gravity, \( D \) is the flexural rigidity of the crust and \( k_n \) is the wavenumber. The flexural rigidity can be used to infer a characteristic wavelength (~\( D/g \rho \rho \)) beyond which elastic support is ineffective.

   For Europa, \( D/g \rho \rho \) is (say) \( 10^{16} \) m⁴ and \( \Delta \rho/\rho \) is about 0.1. Plot the deflection (in the space domain) arising from the topographic load shown in 2b.

   d) Upwards continuation. The gravity \( G_n \) due to topography of amplitude \( H_n \) is given by

   \[
   G_n = 2 \pi G \rho H_n e^{-k_n z}
   \]

   where \( \rho \) is the load density, \( G \) is the gravitational constant, \( z \) is the altitude above the surface and \( k_n \) is the wavenumber. Plot the gravity (in the space domain) you’d
predict from the topographic load shown in 2b at altitudes of 10 km and 30 km, taking the load density to be 1000 kgm⁻³.
[Units: 10⁻⁵ ms⁻² is 1 mGal.]

e) Relaxation. For a uniform viscosity fluid, topography of amplitude $H_n$ decays with time according to

$$H_n = H_{n0} e^{-t/\tau_n}$$

where $H_{n0}$ is the initial amplitude and $\tau_n$ is the relaxation timescale given by

$$\tau_n = \frac{2\eta k_n}{\rho g}$$

For ice, $\eta=10^{15}$ Pa s (say). Taking $g=1$ms⁻², plot the topography in the space domain after relaxation has proceeded for 10 and 100 years.

[Note that in reality ice viscosity varies with depth, in which case the formula for $\tau_n$ depends on the e-folding depth of viscosity variation as well as the wavenumber.]