1a. Here I want you to write a simple code to convert gridded data to spherical harmonic coefficients $C_{lm}, S_{lm}$. The algorithm is:

$$C_{lm} = \frac{1}{4\pi} \sum_j \sum_k h_{kj} \cos m \phi_k \tilde{P}_m(\cos \theta_j) \sin \theta_j \Delta \theta \Delta \phi$$

and similarly for $S_{lm}$. Here the elevation at a particular grid-point is $h_{kj}$, with corresponding colatitude and longitude $\theta_j$ and $\phi_k$. The grid spacing is $\Delta \theta$ and $\Delta \phi$.

In case you don’t have a code for doing this, I’ve added a subroutine plmbar.f for the recursive calculation of normalized associated Legendre polynomials to the website.

To check your algorithm, download the gridded topography simple.dat from the website. This has the format latitude, longitude, elevation, with an increment of 15 degrees in latitude and longitude (cell-centred).

This topography was constructed using only two spherical harmonic coefficients, with amplitudes 1 and 2, respectively. What is the degree and order of each of the two coefficients?

b) Now download the gridded topography synthetic_europa_l3-360-2.dat from the website and use your code to calculate $C_{lm}$ and $S_{lm}$ up to $l=m=50$. The coefficients up to $l=2$ should be zero. This topography has the format latitude, longitude, elevation in km. The increment in lat and long is 0.5 degrees (cell-centered).

Write down the $l=3$ and $l=4$ coefficients so that I can compare them with my values. Be careful about normalizations.

c) Plot the power spectrum of this synthetic topography as a function of degree.

d) Now apply a window to the global topography, determine the new spherical harmonic coefficients and hence plot the resulting power spectrum.

The window is a spherical cap of radius 45°, centred at 20° long, 10° latitude. All topography outside this region should be set to zero. The easiest way to do this is to take each original grid point, calculate its distance from the centre of the window using spherical geometry (the Vincenty formula is useful), and then set the elevation value to zero if the distance exceeds 45°.

Explain why this power spectrum looks different to the one plotted in c).
\textbf{e)} The coefficient of the radial gravity (not the potential) arising from surface topography is given for a particular topographic coefficient $C_{lm}$ by

$$C_{lm}^r = 4\pi G \rho \frac{l + 1}{2l + 1} \left( \frac{1}{1 + \frac{z}{R}} \right)^{l+2} C_{lm} \quad \text{and similarly with } S_{lm}.$$  

Here $\rho$ is the density of the surface material and $z$ is the elevation above the surface (radius $R$). We are assuming $C_{lm}$ has dimensions of length here.

Plot the gravity (in mGal) arising from the original topographic coefficients at an altitude $z=100$ km assuming that the surface density is 900 kgm$^{-3}$ and the radius is 1565 km. One mGal=10$^{-5}$ ms$^{-2}$.

\textbf{2. Triaxial ellipsoid}

Many planetary objects can be approximated as triaxial ellipsoids with axes $a=R+\Delta a$, $b=R+\Delta b$, $c=R+\Delta c$ where the mean radius is $R^2=(a^2+b^2+c^2)/3$.

a) We can use the degree-two coefficients $C_{20}$ and $C_{22}$ to describe the departure from spherical shape, that is $\Delta a, \Delta b$ and $\Delta c$. In terms of colatitude $\theta$ and longitude $\phi$, the axes $a,b,c$ have coordinates $(\pi/2, 0)$, $(\pi/2, \pi/2)$ and $(0,0)$.

Write down expressions for $\Delta a, \Delta b, \Delta c$ and $\Delta a+\Delta b+\Delta c$ in terms of $C_{20}$ and $C_{22}$. Here it is best to use the \textit{non-normalized} Legendre functions.

b) A uniform density triaxial ellipsoid will have non-zero degree-2 potential coefficients, with the relationship given by

$$C_{lm}^g = \frac{3}{2l+1} \frac{1}{R} C_{lm}$$

Write down the two potential coefficients $C_{20}^g$ and $C_{22}^g$ in terms of $\Delta a, \Delta b$ and $\Delta c$.

c) Yoder (1995, Global Earth Physics handbook) gives a higher-order relationship as follows:

$$C_{20}^g = \frac{1}{5R^2} \left( c^2 - \frac{1}{3} (a^2 + b^2) \right)$$

$$C_{22}^g = \frac{1}{20R^2} (b^2 - a^2)$$

Verify that these results are consistent with yours in the limit of small $\Delta a, \Delta b, \Delta c$. 