Spherical Harmonics

Topics/Concepts
Orthogonality
Normalization!
Associated Legendre functions
Parseval’s theorem
Gibbs phenomenon
Aliasing, spectral leakage
Windows, tapering, spherical cap
Spectral resolution – spatial resolution tradeoff
Upwards attenuation, flat plate formula

Equations
Zonal field (spherical, axial-symmetric, \( \mu = \cos \theta \)):

\[
a_n = \frac{2n+1}{2} \int_{-1}^{1} f(\mu) P_n(\mu) d\mu \quad P_n(\mu) \text{ are Legendre polynomials} \]

General spherical harmonics:

\[
\sum_{l,m} \left( \begin{array}{c} C_{lm} \cos m\phi + S_{lm} \sin m\phi \end{array} \right) P_{lm}(\theta) \]

Approximate wavelength: \( \lambda \approx \frac{2\pi}{l} \)

“4\pi” normalization (used here):

\[
\sum_{l,m} \left( \begin{array}{c} C_{lm} \cos m\phi + S_{lm} \sin m\phi \end{array} \right) P_{lm}(\cos \theta) \]

Power at degree-\( l \):

\[
P(l) = \sum_{m} C_{lm}^2 + S_{lm}^2
\]

Parseval’s theorem:

\[
\frac{1}{4\pi} \int h_{\phi\phi}^2 \sin \theta d\theta d\phi = \sum_{l} P(l) = \sum_{l} \sum_{m} C_{lm}^2 + S_{lm}^2
\]

(Using 4\pi normalization)

External gravitational potential:

\[
U(r) = \frac{GM}{r} \sum_{l} \sum_{m} \left( \frac{2l+1}{(l-m)!} \right) C_{lm} \cos m\phi + S_{lm} \sin m\phi P_{lm}(\cos \theta)
\]

Magnetic potential (internal sources):

\[
V(r) = R_0 \sum_{l} \sum_{m} \left( \frac{2l+1}{(l-m)!} \right) \left( g_{lm} \cos m\phi + h_{lm} \sin m\phi \right) P_{lm}(\cos \theta)
\]

Magnetic power \( P_n \):

\[
P_n = \frac{1}{4\pi R_0^2} \int \mathbf{B} \cdot \mathbf{B} R_0^2 \sin \theta d\theta d\phi = (n+1) \sum_{m} g_{lm}^2 + s_{lm}^2
\]
References

Legendre Polynomials $P_n(\theta)$

$$f(\theta) = a_1 P_1(\theta) + a_2 P_2(\theta) + \cdots$$

Legendre Polynomials

<table>
<thead>
<tr>
<th>$n$</th>
<th>$P_n(\mu)$</th>
<th>$P_n(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\mu$</td>
<td>$\cos \theta$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2}(3\mu^2 - 1)$</td>
<td>$\frac{1}{4}(3\cos 2\theta + 1)$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{2}(5\mu^3 - 3\mu)$</td>
<td>$\frac{1}{8}(5\cos 3\theta + 3\cos \theta)$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$</td>
<td>$\frac{1}{64}(35\cos 4\theta + 20\cos 2\theta + 9)$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{8}(63\mu^5 - 70\mu^3 + 15\mu)$</td>
<td>$\frac{1}{128}(63\cos 5\theta + 35\cos 3\theta + 30\cos \theta)$</td>
</tr>
</tbody>
</table>

Here $f(\theta)=1$ for $\theta<\theta_0$, $f(\theta)=0$ otherwise
Legendre Polynomials and Gibbs Overshoot

Fig. 6.3. A few low-degree Legendre functions. (a) Functions $P_0(\mu)$ through $P_6(\mu)$ are shown on the interval $-1 \leq \mu \leq 1$. (b) Function $P_6(\mu)$ is shown along the circumference of a circle; grey and white zones indicate areas where the function would be positive or negative, respectively, if wrapped around a sphere.

Fig. 6.5. Approximation of the discontinuous function shown in Figure 6.4 with the sum of three Legendre polynomials. (a) Unweighted Legendre functions of degree 1, 2, and 3; (b) the weighted sum of the three Legendre functions.
Associated Legendre Polynomials

\[ P_{11} = \sin \theta \]
\[ P_{21} = \frac{1}{2} \sin 2\theta \]
\[ P_{31} = \frac{1}{4} \sin \theta (5 \cos 2\theta + 3) \]
\[ P_{22} = 3 \sin^2 \theta \]
\[ P_{32} = \frac{15}{2} \sin \theta \sin 2\theta \]
\[ P_{33} = 15 \sin^3 \theta \]

Visualizing Spherical Harmonics

Pairs of images for \(11, 21, 22, 31, 32, 33\) show \(\sin m\phi\) and \(\cos m\phi\), respectively.