Last Week – Shapes, geoid, topography

• How do we measure shape/topography?
  – GPS, altimetry, stereo, photoclinometry, limb profiles, shadows

• What is topography referenced to?
  – Usually the geoid (an equipotential)
  – Sometimes a simple ellipsoid (Venus, Mercury)

• What controls the global shape of a planet/satellite?
  What does that shape tell us?
  – Rotation rate, density, (rigidity)
  – Fluid planet \( f \sim \Omega^2 a/2g \) Satellite \( f \sim 5\Omega^2a/g \)

• What does shorter-wavelength topography tell us?
  – Hypsometry, roughness, elastic thickness?
This Week – Rheology

- Definitions: Stress, strain and strength
- How do materials respond to stresses?: elastic, brittle and viscous behaviour
- What loads does topography impose?
- Elastic, viscous and brittle support of topography
The relation between the stress applied to a material and the resulting deformation is called *rheology*.

- **Elastic**: Describes materials that *return to their rest shape* after an applied stress.
- **Plastic**: Describes materials that *permanently deform* after a sufficient applied stress.
- **Viscous**: Describes materials that *flow* in response to an applied stress (zero long-term deformation).

Most materials behave as some combination of these.

Important particular cases:
- **Brittle**: A specific kind of plastic deformation
- **Viscoelastic**: A combination of viscous and elastic
- **Viscoplastic**: A combination of viscous and plastic
Stress ($\sigma$) and Strain ($\varepsilon$)

- **Normal stress**: $\sigma = F / A$
  (stress perpendicular to plane)
- **Example**: mass of overburden per unit area = $\rho h$, so pressure (stress) = $\rho gh$
- **Shear stress**: $\sigma = F / A$
  (stress parallel to plane)
- **Normal strain** $\varepsilon = \Delta L / L$ (dimensionless)
- **Shear strain** $\varepsilon = \Delta L / L$ (shear angle)
- **In three dimensions** $\Delta$, the fractional change in volume, $= \Delta V / V = \varepsilon_x + \varepsilon_y + \varepsilon_z$
Elasticity and Hooke’s Law

• Materials which are below about 70% of their melting temperature (in K) typically behave in an elastic fashion.

• In the elastic regime, stress is proportional to strain (Hooke’s law):

\[ \varepsilon = \frac{\sigma}{E} \]

• The constant of proportionality \( E \) is Young’s modulus.

• For shear strain we use the shear modulus \( G \):

\[ \varepsilon_{xy} = \frac{\sigma_{xy}}{2G} \]

• These moduli tell us how resistant to deformation a particular material is (how much strain for a given stress).

• Typical values of Young’s modulus are \( 10^{11} \) Pa (for rock) and \( 10^{10} \) Pa (for ice).
Poisson’s ratio

- Unconfined materials will expand perpendicular to the applied stress
- Amount of expansion is given by Poisson’s ratio $\nu$
- What is an example of a material with a negative value of $\nu$?

- A material with $\nu=1/2$ is incompressible. What does this mean?
- Geological materials generally have $\nu = 1/4$ to $1/3$
- We can obtain a useful relationship between Young’s modulus $E$ and shear modulus $G$:

\[
G = \frac{E}{2(1+\nu)}
\]
Resolving stresses

In general, both normal \((\sigma_{xx}, \sigma_{yy})\) and shear \((\sigma_{xy})\) stresses are acting. We can resolve these stresses onto any plane. Stress = force/area. So we care about the angle at which the force is acting, and the area over which it is acting.

\[
\sigma_n = \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + 2\sigma_{xy} \sin \theta \cos \theta
\]

\[
\sigma_s = \sigma_{yy} \cos \theta \sin \theta - \sigma_{xx} \cos \theta \sin \theta + \sigma_{xy} \cos 2\theta
\]

Why are \(\sigma_{xy}\) and \(\sigma_{yx}\) the same?

Do these equations make sense in the appropriate limits?

We can use these two equations to determine a principal stress direction in which there are no shear stresses acting:

\[
\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}
\]
Plastic response to a load

- *Irreversible* deformation in response to applied stress
- Once stress is released, material stays deformed
- Usually occurs above some elastic limit
“Strength”

- The maximum load (stress) that can be supported

Different kinds of strength:
- Yield stress \((Y)\) – onset of plastic deformation \((T\)-dependent\)
- Cohesive strength \((Y_0)\) – rock resistance to tensile failure
  (depends on flaws in the material; \(\sim Y\) for single crystals)

<table>
<thead>
<tr>
<th>Material</th>
<th>Cohesion (Y_0) (MPa)</th>
<th>Yield Stress (Y) (GPa)</th>
<th>Shear modulus (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quartzite</td>
<td>60</td>
<td>0.61</td>
<td>44</td>
</tr>
<tr>
<td>Ice</td>
<td>(\sim 1)</td>
<td>0.2-1.0 (\text{at 77-115 K})</td>
<td>3.4</td>
</tr>
<tr>
<td>Iron</td>
<td>Same as (Y)</td>
<td>0.1-1.0</td>
<td>82</td>
</tr>
<tr>
<td>Forsterite</td>
<td>Same as (Y)</td>
<td>1.1</td>
<td>81</td>
</tr>
</tbody>
</table>

Materials with high shear modulus tend to have high yield stresses
Stresses due to topography

“Jeffrey’s theorem”: Minimum stress difference required to support a load is \(~1/3\) of the imposed load, irrespective of the load geometry and support mechanism.

Region of large stresses typically concentrated in an area with dimensions comparable to the \textit{width} of the load.

Strictly speaking, it is the stress \textit{difference} that is being considered here i.e. \(\sigma_1-\sigma_3\).
Maximum mountain height

\[ \sigma_{\text{max}} \approx \frac{\rho gh}{3} \]

\[ h_{\text{max}} \approx \frac{9}{4\pi} \frac{Y}{G\rho R\rho_c} \]

Note that here \( G \) is the gravitational constant.

Theory works well for Mars and the Earth, but not the Moon. Why?

(Y=100 MPa here)
Brittle (frictional) behaviour

- For pre-existing fractures in rock (or ice), there is a linear relationship between the normal stress ($\sigma$) and the shear stress ($\tau$) required to cause motion:

$$\tau = c + f\sigma$$

- This is known as Byerlee’s law

$f$ is a friction coefficient

$f=0.6$ for most materials
Viscous behaviour

• At temperatures > ~70% of the melting temperature (in K), materials will start to flow (ductile behaviour)
• E.g. ice (> -80°C), glass/rock (> 700°C)
• Cold materials are elastic, warm materials are ductile
• The basic reason for flow occurring is that atoms can migrate between lattice positions, leading to much larger (and permanent) deformation than the elastic case
• The atoms can migrate because they have sufficient (thermal) energy to overcome lattice bonds
• This is why flow processes are a very strong function of temperature
Elasticity and Viscosity

• Elastic case – strain depends on stress (Young’s mod. $E$)
  \[ \varepsilon = \sigma / E \]

• Viscous case – strain rate depends on stress (viscosity $\eta$)
  \[ \dot{\varepsilon} = \sigma / \eta \]

• We define the (Newtonian) viscosity as the stress required to cause a particular strain rate – units Pa s

• Typical values: water $10^{-3}$ Pa s, basaltic lava $10^4$ Pa s, ice $10^{14}$ Pa s, mantle rock $10^{21}$ Pa s

• Viscosity is a macroscopic property of fluids which is determined by their microscopic behaviour
Newtonian vs. Non-Newtonian Viscosity

- Strain rate depends on applied stress:
  \[ \dot{\varepsilon} \propto \sigma^n \]

- We can define an (effective) viscosity:
  \[ \eta = \frac{\sigma}{\dot{\varepsilon}} \]

- For a Newtonian material, \( n=1 \) and viscosity is independent of stress
Viscosity

\[ \dot{\varepsilon} = A \sigma^n d^{-p} \exp\left(-\frac{Q}{RT}\right) \]

\[ \eta_{\text{eff}} = \frac{\sigma}{\dot{\varepsilon}} = \frac{d^p \exp\left(\frac{Q}{RT}\right)}{A \sigma^{n-1}} \]

\[ R \text{ is gas constant} \]
\[ Q \text{ is activation energy} \]

Ice.
Goldsby & Kohlstedt (2001)

Different creep mechanisms operate under different conditions

Higher \( T \)
Higher strain rate
Smaller grain size \((d)\)
Higher strain rate

Higher stress
Higher strain rate

Higher strain rate

You can use the graph to read off the viscosity. E.g. this value is \(10^{12} \text{ Pa s}\)
Lithosphere and Below

Aesthenosphere (↔ Flows →)

Lithosphere (Rigid)

Thermal boundary layer: Conduction

$T_s$

$q_s = k \left. \frac{dT}{dz} \right|_0$

$\delta$

$z$

$\frac{dT}{dz}$ (adiabatic)

$T_m / 2$
Strength Envelopes

- Materials respond to stresses by brittle failure or ductile creep
- The *lower* of the two stresses controls what happens
- At low temperatures -> brittle deformation
- At high temperatures -> ductile creep
- So we can construct strength envelopes:

  ![Strength Envelopes Diagram](image)

- A larger area under the curve indicates a “stronger” lithosphere

  These strength-envelopes depend mainly on temperature gradient and gravity (why?)

- Why does the Moon have a deep brittle-ductile transition?
Why are planets round?

• Internal stresses exceed the “strength” of the material

• Maximum shear stress a rubble pile can withstand: \( \sigma_{\text{max}} \approx f \rho gd \)

• Stresses generated by topography: 
  \[ \sigma_{\text{topo}} \approx \frac{\rho gh}{3} \]

• Balance the two against each other:
  \[ h_{\text{max}} \approx 3 fR \]

Here we’ve taken \( d=R \)
Why are planets round?

- Rubble pile (friction): \( \frac{h_{\text{max}}}{R} \approx 3f \)
- Solid planet (yield strength): \( \frac{h_{\text{max}}}{R} \approx \frac{9}{4\pi} \frac{Y}{G \rho R^2 \rho_c} \)

![Graph showing the relationship between \( h_{\text{max}}/R \) and mean radius, with data points for rocky and icy planets. The slope is -2, indicating a quadratic relationship.]
Iapetus

Castillo-Rogez et al. 2007

Radius = 735 km, density = 1.08 g/cc
Spin period = 79 days (slow!)

(a-c) should be about 10m
Actual (a-c) is about 35 km!

• What’s going on?
• How large are the stresses involved?
• What does this tell us about Iapetus’s evolution?
Supporting topography

- Topography imposes loads on the near-surface
- These loads can be supported by a variety of mechanisms
- Which mechanism operates tells us about the near-surface (and deeper) structure of the planet
- See EART162 for more details

A positive load (Tohil Mons, Io, 6 km high)  A negative load (Herschel crater, Mimas, 12 km deep)
Airy Isostasy

- In the case of no elastic strength, the load is balanced by the mantle root: \( h \rho_c = r(\rho_m - \rho_c) \)
- This also means that there are no lateral variations in pressure beneath the crustal root

- So crustal thickness contrasts \((\Delta t_c = h + r)\) lead to elevation contrasts \((h)\):
  \[
  h = \frac{(\rho_m - \rho_c)}{\rho_m} \Delta t_c
  \]
- Note that the elevation is independent of the background crustal thickness \(t_c\)
Pratt Isostasy

- Similar principle – pressures below some depth do not vary laterally
- Here we do it due to variations in density, rather than crustal thickness

\[ h = \frac{(\rho_2 - \rho_1)}{\rho_2} t_c \]

- What’s an example of where this mechanism occurs on Earth?
Gravity Effects

• Because there are no lateral variations in pressure beneath a certain depth, that means that the total mass above this depth does not vary laterally either.
• So what sort of gravity anomalies should we see?
• Very small ones!

Uncompensated load: \( \Delta g = 2\pi \rho_c G h \)
Compensated load: \( \Delta g \approx 0 \)

(NB there will actually be a small gravity anomaly and edge effects in this case)

So we can use the size of the gravity anomalies to tell whether or not surface loads are compensated.
Example - Mars

- The southern half of Mars is about 3 km higher than the northern half (why?)
- But there is almost no gravity anomaly associated with this “hemispheric dichotomy”
- We conclude that the crust of Mars here must be compensated (i.e. weak)
- Pratt isostasy? Say $\rho_1=2700 \text{ kgm}^{-3}$ (granite) and $\rho_2=2900 \text{ kgm}^{-3}$ (basalt). This gives us a crustal thickness of 45 km
Mars (cont’d)

• On the other hand, some of the big volcanoes (24 km high) have gravity anomalies of 2000-3000 mGal
• If the volcanoes were sitting on a completely rigid plate, we would expect a gravity anomaly of say $2.9 \times 24 \times 42 \approx 2900$ mGal
• We conclude that the Martian volcanoes are almost uncompensated, so the crust here is very rigid
• Remember that what’s important is the strength of the crust at the time the load was emplaced – this may explain why different areas have different strengths

\[ \Delta g = 2\pi \rho_c G h \]
Lunar Mascons

- Gravity highs associated with topography *lows* (basins)
- Very puzzling!
- Some combination of initial *superisostatic* uplift of Moho and later surface loading by dense mare basalts after lithosphere cooled
Flexure

• So far we have dealt with two end-member cases: when the lithosphere is completely rigid, and when it has no strength at all (isostasy)

• It would obviously be nice to be able to deal with intermediate cases, when a load is only partly supported by the rigidity of the lithosphere

• I’m not going to derive the key equation – see EART162 for more details

• We will see that surface observations of deformation can be used to infer the rigidity of the lithosphere

• Measuring the rigidity is useful because it is controlled by the thermal structure of the subsurface
Flexural Stresses

• In general, a load will be supported by a combination of elastic stresses and buoyancy forces (due to the different density of crust and mantle)
• The elastic stresses will be both compressional and extensional (see diagram)
• Note that in this example the elastic portion includes both crust and mantle
Flexural Parameter (1)

- Consider a line load acting on a plate:

- Except at $x=0$, load = 0 so we can write:

$$D \frac{d^4 w}{dx^4} + (\rho_m - \rho_w)gw = 0$$

- Boundary conditions for an unbroken plate are that $dw/dx = 0$ at $x=0$ and $w \to 0$ as $x \to \infty$

- The solution is $w = w_0 \exp(-x/\alpha)(\cos \frac{x}{\alpha} + \sin \frac{x}{\alpha})$

- Here $\alpha$ is the flexural parameter

$$\alpha = \left[\frac{4D}{g(\rho_m - \rho_w)}\right]^{\frac{1}{4}}$$

$D$ is the flexural rigidity, $D = ET_e^3/12(1-\nu^2)$
Flexural Parameter (2)

- Flexural parameter $\alpha=\left(\frac{4D}{g(\rho_m-\rho_w)}\right)^{1/4}$
- It is telling us the *natural wavelength* of the elastic plate
- E.g. if we apply a concentrated load, the wavelength of the deformation will be given by $\alpha$
- Large $D$ gives long-wavelength deformation, and v.v.
- If the load wavelength is $\gg \alpha$, then the plate is *weak* and deformation will approach the isostatic limit
- If the load wavelength is $\ll \alpha$, then the plate is *rigid*, deformation will be small and have a wavelength $\sim \alpha$
- If we can measure a flexural wavelength, that allows us to infer $\alpha$ and thus $D$ or $T_e$ directly. This is useful!
Example: Serenity Chasma, Charon

By eye, $\alpha \sim 20$ km (assuming that we are seeing a flexural response!). This implies $T_e \sim 2.5$ km. Numerical fits give a very similar answer.

Flexural fits courtesy Jack Conrad
Dynamic topography

- Another way of supporting topography is from viscous stresses due to motion in the mantle.
- This requires convection to be occurring.
- It is a long-wavelength effect, operating on Earth, Venus and (maybe) Mars and Io.
Viscous relaxation

\[ w = w_0 \exp(-t/\tau) \]

\[ \tau \sim \frac{\eta}{\rho g L} \]

Hudson’s Bay deglaciation:
\( L \sim 1000 \text{ km}, \ \tau = 2.6 \text{ ka} \)

So \( \eta \sim 2 \times 10^{21} \text{ Pa s} \)

So we can infer the viscosity of the mantle

A longer wavelength load would sample the mantle to greater depths – higher viscosity
Relaxed craters

- Provide a probe of subsurface viscosity (and thus temperature) structure
- Bigger craters generally more relaxed (why?)

Ganymede shows a mixture of relaxed and unrelaxed craters – how come?

Iapetus has big basins which are incompletely relaxed

Dombard & McKinnon (2006)

300 km

Robuchon et al. (2011)
Viscoelasticity

- Real geological materials behave as elastic solids at short periods, but viscous fluids at long periods
- E.g. Earth’s mantle responds elastically to seismic waves (~1s), but convects like a fluid (~Myr)
- A material which has this property is viscoelastic
- A Maxwell material has a relaxation time: $\tau_M = \frac{\eta}{G}$
  (Here $G$ is shear modulus)

- At timescales $>> \tau_M$, the material will be a fluid
- At timescales $<< \tau_M$, the material will be elastic
- What is the Maxwell time of the Earth’s mantle?
Summary – Rheology

- **Definitions**: Stress, strain and strength
  - strength = maximum stress supported, $\sigma = F/A$, $\varepsilon = \Delta L/L$
- **How do materials respond to stresses?**: elastic, brittle and viscous behaviour
  - Elastic $\sigma = E \varepsilon$
  - Brittle $\tau = c + f \sigma$
  - Viscous $\sigma = \eta \frac{d\varepsilon}{dt}$
- **What loads does topography impose?**
  - $\sigma \sim 1/3 \rho gh$
- **Elastic, viscous and brittle support of topography**
  - Flexure, $\alpha = (4D/\Delta \rho g)^{1/4}$
  - Viscous relaxation and dynamic support
  - Role of yield stress $Y$ and friction coefficient $f$ in controlling topography on large and small bodies, respectively
• What gravity signals are associated with $C=1$ and $C=0$?
• How would the curves move as $T_e$ changes?

So by measuring the ratio of gravity to topography (*admittance*) as a function of wavelength, we can infer the elastic thickness of the lithosphere remotely.
Viscosity in action

- Definition of viscosity is stress / strain rate
- For a typical terrestrial glacier \( \mu = 10^{14} \) Pa s.
- Typical stresses driving flow are \( \sim 1 \) MPa (why?)

\[
\text{Strain rate} = \frac{\text{stress}}{\text{visc}} = 10^{-8} \text{ s}^{-1}
\]
Centre-line velocity \( \sim 10^{-5} \) m s\(^{-1}\) \( \sim 0.3 \) km per year

(Velocity profile is not actually linear because of non-Newtonian nature of ice)

- Temperature-dependence of viscosity is very important. E.g.
  - Do glaciers flow on Mars?
  - How can the Earth’s mantle both convect and support surface loads simultaneously?