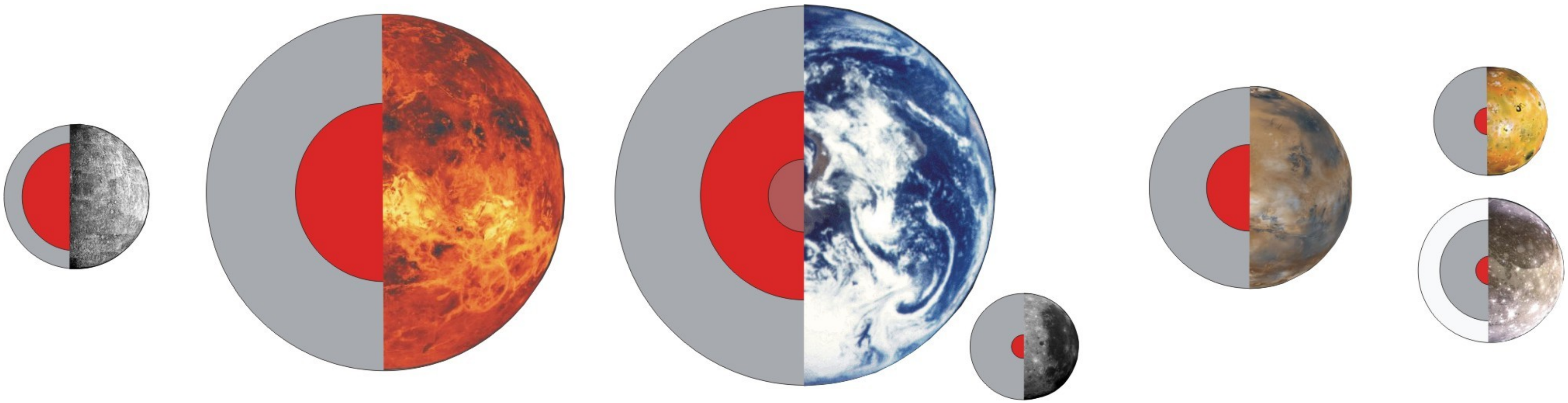
EART162: PLANETARY INTERIORS



Francis Nimmo

Last Week

- Fluid dynamics can be applied to a wide variety of geophysical problems
- Navier-Stokes equation describes fluid flow:

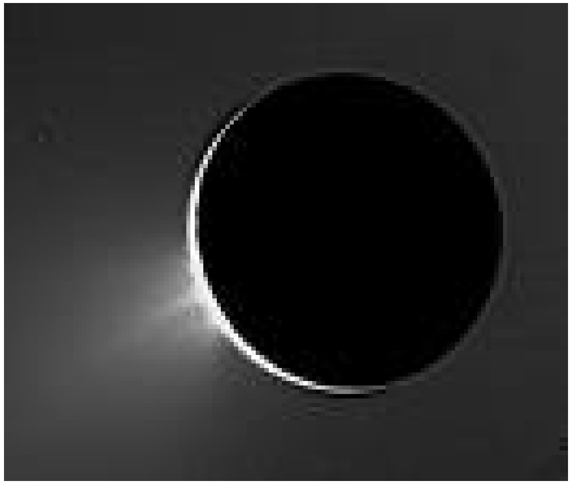
$$\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right] = \eta\underline{\nabla^2 u} - \underline{\nabla}P + \Delta\rho g\,\underline{\hat{y}}$$

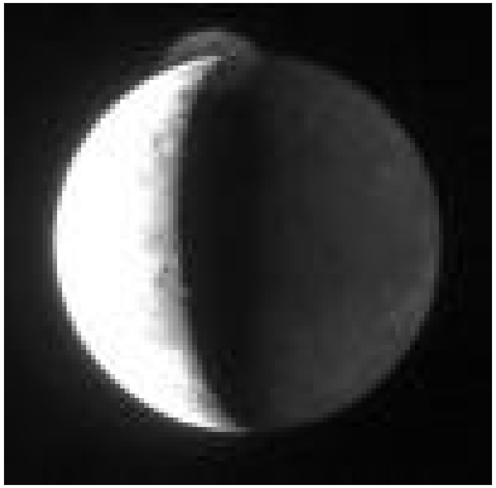
- Post-glacial rebound timescale: $\tau \sim \frac{\eta}{\rho g L}$
- Behaviour of fluid during convection is determined by a single dimensionless number, the Rayleigh number *Ra*

$$Ra = \frac{\rho g \alpha \Delta T d^3}{\kappa\eta}$$

This Week – Tides

- Planetary tides are important for two reasons:
  - They affect the orbital & thermal evolution of satellites
  - We can use tidal effects to infer satellite moments of inertia (and thus internal structure)





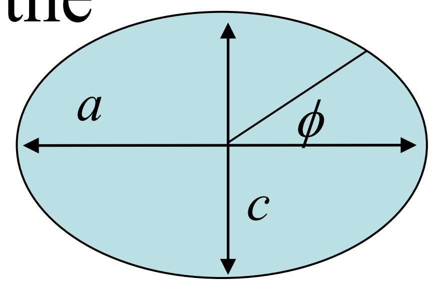
Recap – planetary shapes

- For a rotationally flattened planet, the potential is:

$$U = -\frac{GM}{r} + \frac{GMa^2 J_2}{2r^3}[3\sin^2\phi - 1] - \tfrac{1}{2}\omega^2 r^2 \cos^2\phi$$

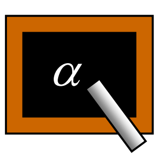
- This is useful because a **fluid** will have the *same potential* everywhere on its surface

- Let's equate the polar and equatorial potentials for our rotating shape, and let us also define the ellipticity (or flattening): $f = \frac{a-c}{a}$



- After a bit of algebra, we end up with:

Note approximate!



$$f \approx \frac{3}{2}J_2 + \frac{1}{2}\frac{a^3\omega^2}{GM}$$

Remember that this only works for a *fluid* body!

Planetary shapes cont'd

$$f \approx \frac{3}{2} J_2 + \frac{1}{2} \frac{a^3 \omega^2}{GM}$$

- The flattening f depends on how fast the planet spins and on J_2 (which also depends on the spin rate)
- We can rewrite this expression:

$$f \approx \frac{1}{2} \frac{a^3 \omega^2}{GM} h_{2f}$$

What does this term represent?

- Where h_{2f} is the (fluid) Love number and which tells us *how much the planet is deformed by rotation*
- So measuring f gives us h_{2f}.
- What controls the fluid Love number?

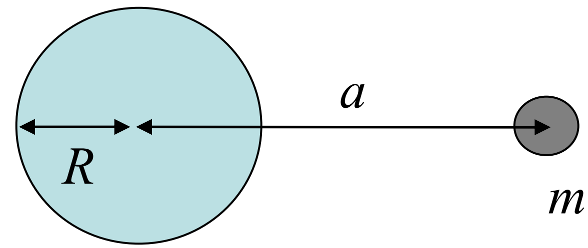
Darwin-Radau (again)

- The Darwin-Radau relationship allows us to infer the MoI of a fluid body given a measurable quantity like J_2 or h_{2f} or the flattening
- We can write it many ways, but here's one:

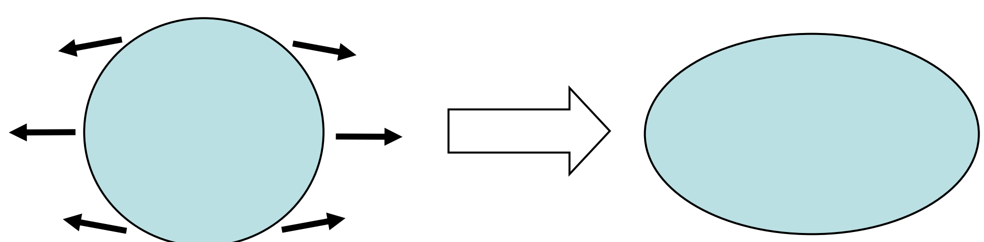
$$\frac{C}{MR^2} = \frac{2}{3}\left[1 - \frac{2}{5}\left(\frac{5}{h_{2f}} - 1\right)^{1/2}\right]$$

- A uniform body has h_{2f}=5/2 and C/MR^2=0.4
- A more centrally-condensed body has a lower h_{2f} and a lower C/MR^2
- So we can measure f, which gives us h_{2f}, which gives us C/MR^2. We can do something similar with satellites . . .

Tides (1)



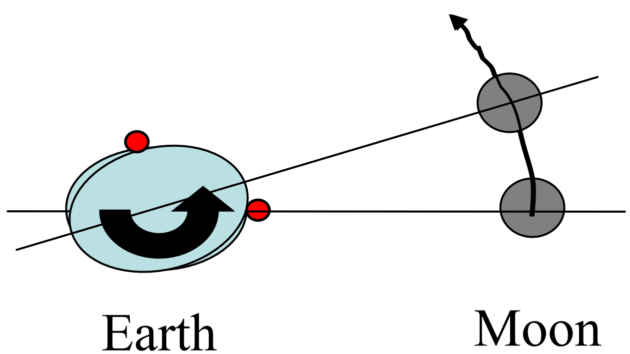
- Body as a whole is attracted with an acceleration = Gm/a^2
- But a point on the far side experiences an acceleration = $Gm/(a+R)^2$
- The *net* acceleration is $2GmR/a^3$ for $R<<a$
- On the near-side, the acceleration is positive, on the far side, it's negative
- For a deformable body, the result is a symmetrical tidal bulge:



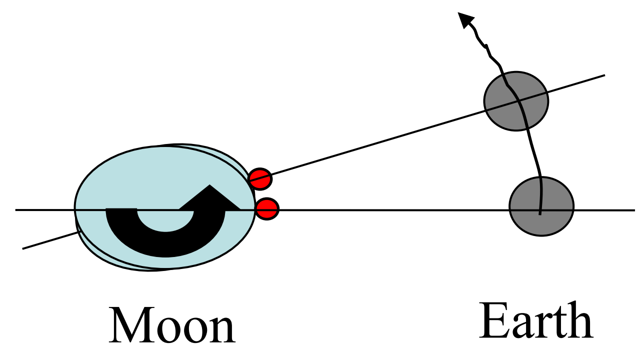
Tides (2)

- It is often useful to think about tidal effects in the frame of reference of the tidally-deforming body

E.g. tides raised on Earth by Moon (Earth rotates faster than Moon orbits, you feel the tidal bulge move past you)

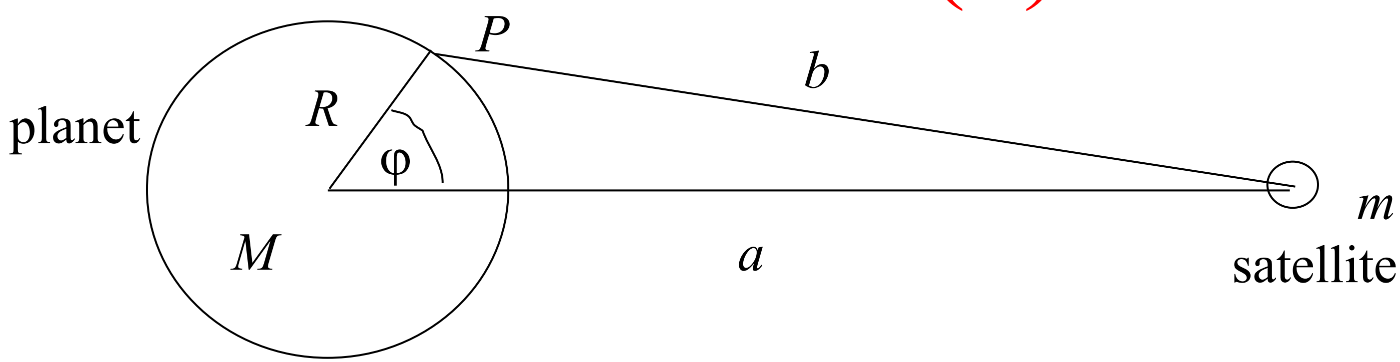


E.g. tides raised on the Moon by Earth (Moon rotates as fast as the Earth appears to orbit, the bulge is (almost) fixed)



- If the Moon's orbit were circular, the Earth would appear fixed in space and the tidal bulge would be static

Tides (3)



- Tidal potential at *P* $$V = -G\frac{m}{b}$$ (recall acceleration = $-\nabla V$)
- Cosine rule $$b = a\left[1 - 2\left(\frac{R}{a}\right)\cos\varphi + \left(\frac{R}{a}\right)^2\right]^{1/2}$$
- *(R/a)<<1*, so expand square root

$$V = -G\frac{m}{a}\left[1 + \left(\frac{R}{a}\right)\cos\varphi + \left(\frac{R}{a}\right)^2 \frac{1}{2}\left(3\cos^2\varphi - 1\right) + \cdots\right]$$

Constant
=> No acceleration

Mean gravitational acceleration (Gm/a^2)

Tide-raising part of the potential

Tides (4)

- We can rewrite the tide-raising part of the potential as

$$-G\frac{m}{a^3}R^2\frac{1}{2}\left(3\cos^2\varphi-1\right)=-HgP_2(\cos\varphi)$$

- Where $P_2(\cos\varphi)$ is a Legendre polynomial, g is the surface gravity of the planet, and H is the equilibrium tide

$$g=\frac{GM}{R^2}$$

$$H=R\frac{m}{M}\left(\frac{R}{a}\right)^3$$

This is the tide raised *on the Earth* by the Moon m

- Does this make sense? (e.g. the Moon at $60R_E$, M/m=81)
- For a uniform fluid planet with no elastic strength, the amplitude of the tidal bulge is $(5/2)H$
- In the general case, the amplitude of the tidal bulge is $h_{2t}H$, where h_{2t} is the (tidal) Love number
- The tidal Love number depends on the mass distribution of the body and *also* on its rigidity (see next slide)

Effect of Rigidity

- We can write a dimensionless number $\tilde{\mu}$ which tells us how important rigidity μ is compared with gravity:

$$\tilde{\mu} = \frac{19}{2}\frac{\mu}{\rho g R}$$ (g is acceleration, ρ is density)

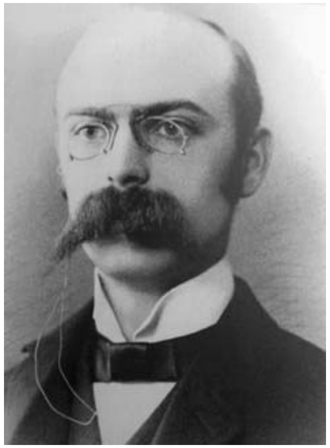
- For Earth, $\mu \sim 10^{11}$ Pa, so $\tilde{\mu} \sim 3$ (gravity and rigidity are comparable)
- For a small icy satellite, $\mu \sim 10^{10}$ Pa, so $\tilde{\mu} \sim 10^2$ (rigidity dominates)
- We can describe the response of the tidal bulge and tidal potential of an elastic body by the tidal Love numbers h_{2t} and k_{2t}, respectively
- For a uniform solid body we have:

$$h_{2t} = \frac{5/2}{1+\tilde{\mu}} \qquad k_{2t} = \frac{3/2}{1+\tilde{\mu}}$$

- E.g. the tidal bulge amplitude d is given by $d = h_{2t} H$ (see last slide)
- If the body is centrally condensed or rigid, then h_{2t} is reduced

Love numbers

- Tidal Love numbers h_{2t} describe the response of the body at tidal frequencies – rigidity may be important
- Fluid Love numbers h_{2f} describe the long-term response of the body (e.g. to rotation) – rigidity not important



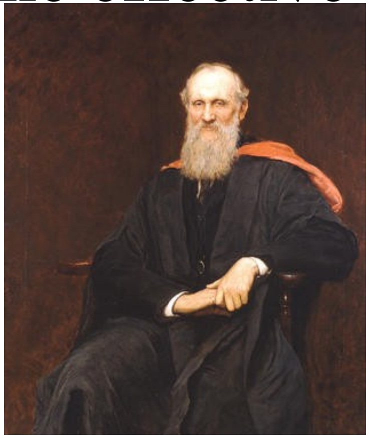
A.E.H. Love

- Example: the solid part of the Earth has a fortnightly tidal amplitude d of about 0.2m. What is the effective rigidity of the Earth?

For Earth, H=0.35m and $d=h_{2t}H$, so h_{2t}=0.6

$h_{2t} = \frac{5/2}{1+\tilde{\mu}}$ So $\tilde{\mu}$ =3 $\tilde{\mu} = \frac{19}{2}\frac{\mu}{\rho g R}$ So **μ=100 GPa**

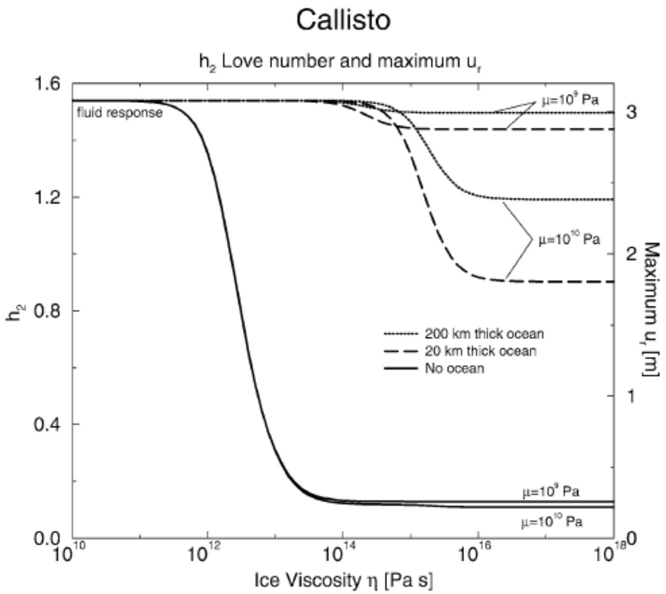
What do we conclude from this exercise?
How do we reconcile this with mantle convection?

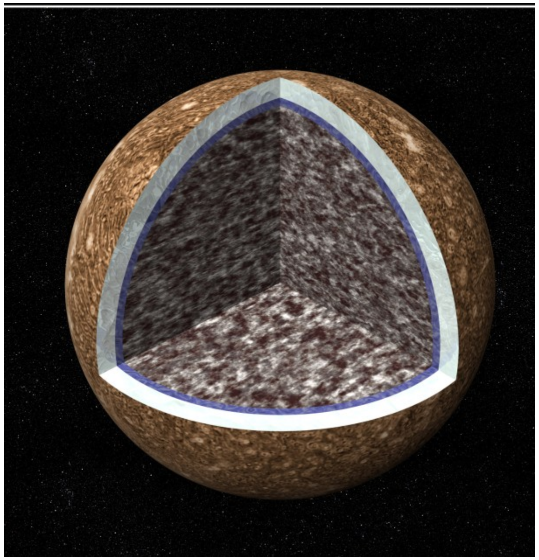


Lord Kelvin

What can the Love number tell us about internal structure?

- Most planets are not uniform bodies
- If the planet has a dense core, then the Love number will be smaller than that of a uniform body with equal rigidity
- If the planet has low-rigidity layers, the Love number will be larger than expected. Why is this useful?





Moore & Schubert 2003

Summary

- The long-term shape (flattening) of a planet is determined by its rotation rate and mass distribution
- The flattening tells us the fluid Love number h_{2f}
- Assuming the planet is fluid, we can use h_{2f} to determine the moment of inertia (via Darwin-Radau)
- The amplitude of the tidal bulge depends on the tidal Love number h_{2t}
- The tidal Love number depends on the mass distribution within the planet and also its rigidity

- We can use similar approaches for satellites

Satellite tides & shapes

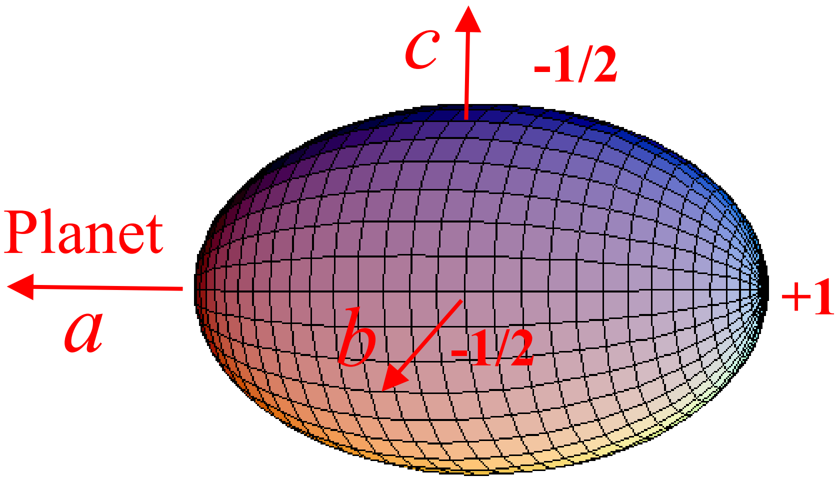
- Most satellites are synchronous – their rotation periods and orbital periods are equal, so the tidal bulge is *static*
- Amplitude of the tidal bulge H_{tid} is $h_{2t} R\ (m/M)\ (R/a)^3$
- Amplitude of the rotational bulge is $h_{2f}\, R\ (R^3 \omega^2 / 3GM)$*
- At long periods, elastic stresses are assumed to relax and so $h_{2t}=h_{2f}$ (no rigidity)
- So the tidal and rotational bulges are in the ratio 3:1

- As long as the satellite behaves like a fluid, we can measure its shape and determine h_{2f} and then use Darwin-Radau to determine its moment of inertia

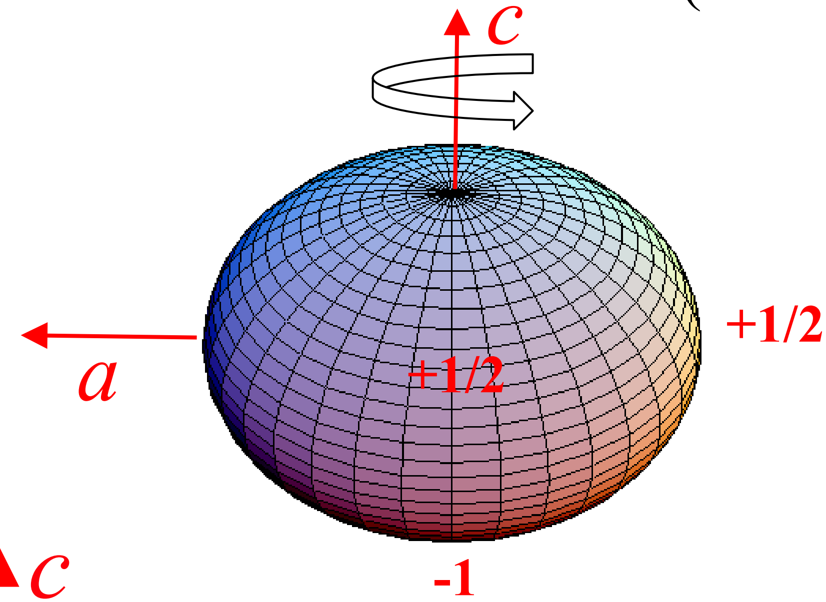
* Factor of 1/3 comes from Legendre function going from -1/2 to +1

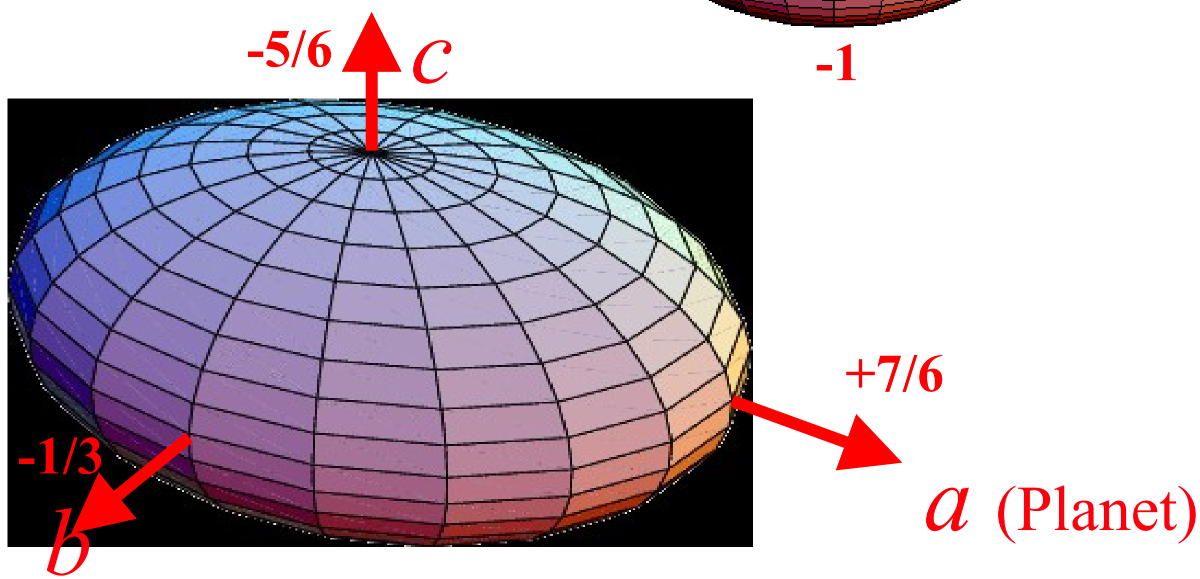
Satellite Shape

Tidal Effect (prolate)



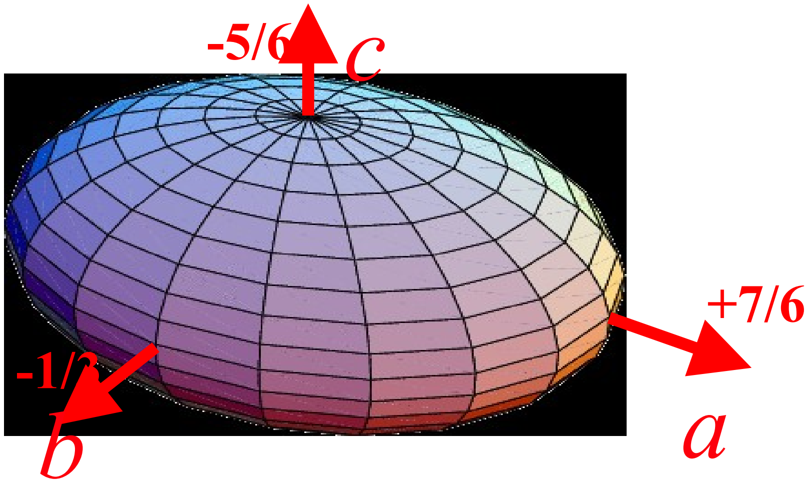
Rotational Effect (oblate)





Dimensions are in units of $h_{2f}H_{tid}$

Satellite Shape (cont'd)



$$a = R\left(1 + \tfrac{7}{6} h_{2f} H_{tid}\right)$$
$$b = R\left(1 - \tfrac{1}{3} h_{2f} H_{tid}\right)$$
$$c = R\left(1 - \tfrac{5}{6} h_{2f} H_{tid}\right)$$

- So for satellites, the flattening f has a different expression to planets: $$f = \frac{a-c}{R} = 2h_{2f}\frac{R^3\omega^2}{GM}$$
- But we can still measure f to infer h_{2f} and the MoI
- For a fluid satellite, we have: $$\frac{b-c}{a-c} = \frac{1}{4}$$
- This provides a very useful *check* on our fluid assumption

Summary

- Satellites are deformed by rotation and tides
- Satellite shape can be used to infer internal structure (as long as they behave like fluids)
- Equivalent techniques exist for gravity measurements

Quantity	Planet	Synch. Sat.
$\frac{a-c}{R}$	$\frac{1}{2}h_{2f}\frac{R^3\omega^2}{GM}$	$2h_{2f}\frac{R^3\omega^2}{GM}$
$\frac{b-c}{a-c}$	1	$\frac{1}{4}$

Only true for *fluid* bodies!

$$\frac{C}{MR^2}=\frac{2}{3}\left[1-\frac{2}{5}\left(\frac{5}{h_{2f}}-1\right)^{1/2}\right]$$

Only true for *fluid* bodies!

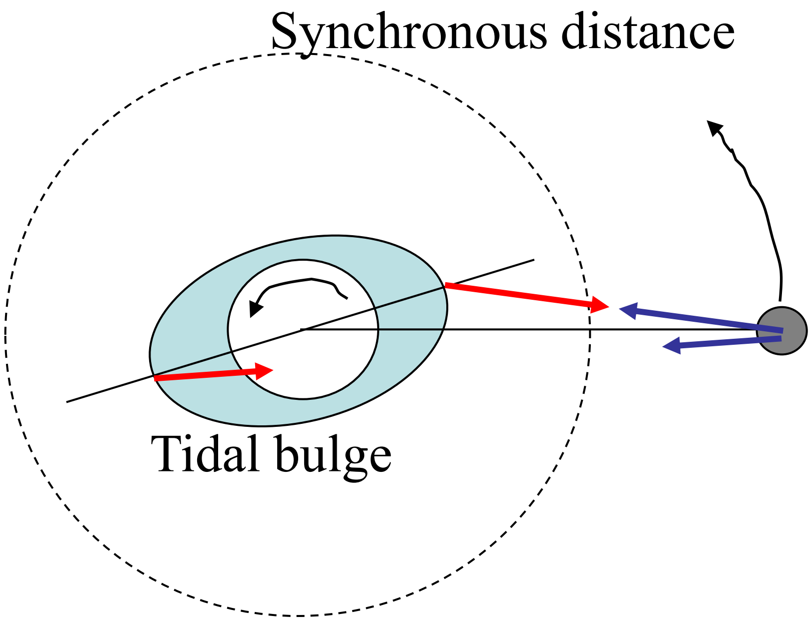
Example - Tethys

- a=540.4 km, b=531.1 km, c=527.5 km
- $(b-c)/(a-c)$ = 0.28 ~ 0.25 so (roughly) hydrostatic
- $(a-c)/R$ = 0.024, $R^3\omega^2/GM$=0.00546 so h_{2f}=2.2
- From Darwin-Radau, C/MR^2=0.366

- What does this imply?
- Tethy's density is 0.973 g/cc. What is this telling us?

Effects of Tides

1) Tidal torques



In the presence of friction *in the primary*, the tidal bulge will be carried ahead of the satellite (if it's beyond the synchronous distance)

This results in a torque on the satellite by the bulge, and vice versa.

The torque on the bulge causes the planet's rotation to slow down

The equal and opposite torque on the satellite causes its orbital speed to increase, and so the satellite moves outwards

The effects are reversed if the satellite is within the synchronous distance (rare – why?)

Here we are neglecting friction in the satellite, which can change things.

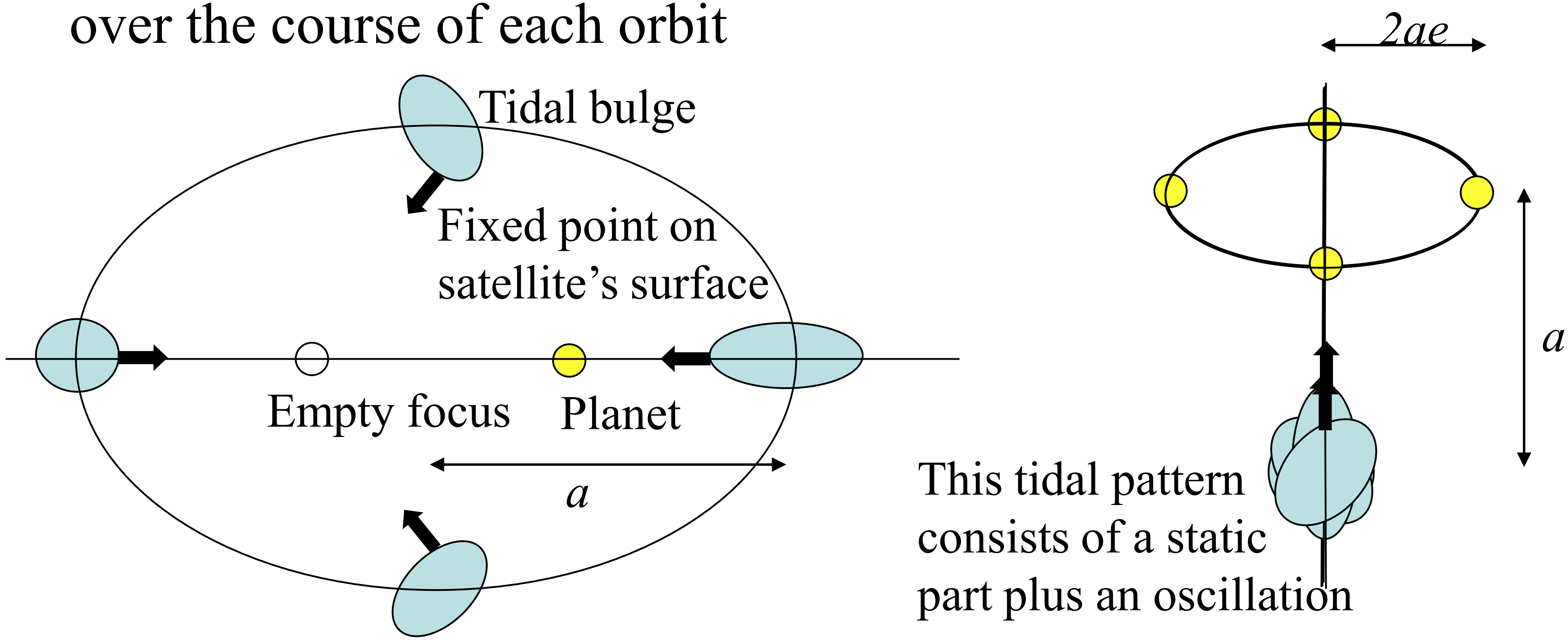
The same argument also applies to the satellite. From the satellite's point of view, the planet is in orbit and generates a tide which will act to slow the satellite's rotation. Because the tide raised by the planet on the satellite is large, so is the torque. This is why most satellites rotate synchronously with respect to the planet they are orbiting.

Tidal Torques

- Examples of tidal torques in action
  - Almost all satellites are in synchronous rotation
  - Phobos is spiralling in towards Mars (why?)
  - So is Triton (towards Neptune) (why?)
  - Pluto and Charon are doubly synchronous (why?)
  - Mercury is in a 3:2 spin:orbit resonance (not known until radar observations became available)
  - The Moon is currently receding from the Earth (at about 3.5 cm/yr), and the Earth's rotation is slowing down (in 150 million years, 1 day will equal 25 hours). What evidence do we have? How could we interpret this in terms of angular momentum conservation? Why did the recession rate cause problems?

Diurnal Tides (1)

- Consider a satellite which is in a synchronous, eccentric orbit
- Both the size and the orientation of the tidal bulge will change over the course of each orbit



- From a fixed point *on the satellite*, the resulting tidal pattern can be represented as a static tide (permanent) plus a much smaller component that oscillates (the diurnal tide)

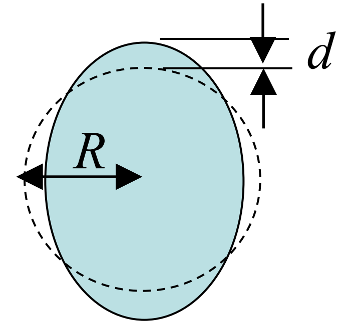
N.B. it's often helpful to think about tides from the satellite's viewpoint

Diurnal tides (2)

- The amplitude of the diurnal tide *d* is *3e* times the static tide (does this make sense?)
- Why are diurnal tides important?
  - Stress – the changing shape of the bulge at any point on the satellite generates time-varying stresses
  - Heat – time-varying stresses generate heat (assuming some kind of dissipative process, like viscosity or friction). NB the heating rate goes as e^2 – we'll see why in a minute
  - Dissipation has important consequences for the internal state of the satellite, and the orbital evolution of the system (the energy has to come from somewhere)
- Heating from diurnal tides dominate the behaviour of some of the Galilean and Saturnian satellites

Tidal Heating (1)

- Recall from Week 5
- Strain depends on diurnal tidal amplitude d



$$\varepsilon \approx d / R$$

- Strain rate depends on orbital period τ
- What controls the tidal amplitude d?
- Power per unit volume P is given by $P \approx \frac{E d^2}{Q R^2 \tau}$
- Here Q is a dimensionless factor telling us what fraction of the elastic energy is dissipated each cycle
- The tidal amplitude d is given by:

$$d = 3 e h_2 H = 3e \frac{5/2}{1+\tilde{\mu}} R \left(\frac{m}{M}\right)\left(\frac{R}{a}\right)^3$$

Tidal Heating (2)

$$P = \frac{Ed^2}{QR^2\tau} = \frac{E}{Q\tau} 9e^2 \frac{25/4}{(1+\tilde{\mu})^2} \left(\frac{m}{M}\right)^2 \left(\frac{R}{a}\right)^6$$

This is not exact, but good enough for our purposes
The exact equation can be found at the bottom of the page

- Tidal heating is a strong function of *R* and *a*
- Is Enceladus or Europa more strongly heated? Is Mercury strongly tidally heated?
- Tidal heating goes as $1/\tau$ and e^2 – orbital properties matter
- What happens to the tidal heating if *e*=0?
- Tidal heating depends on how rigid the satellite is (*E* and μ)
- What happens to *E* and μ as a satellite heats up, and what happens to the tidal heating as a result?

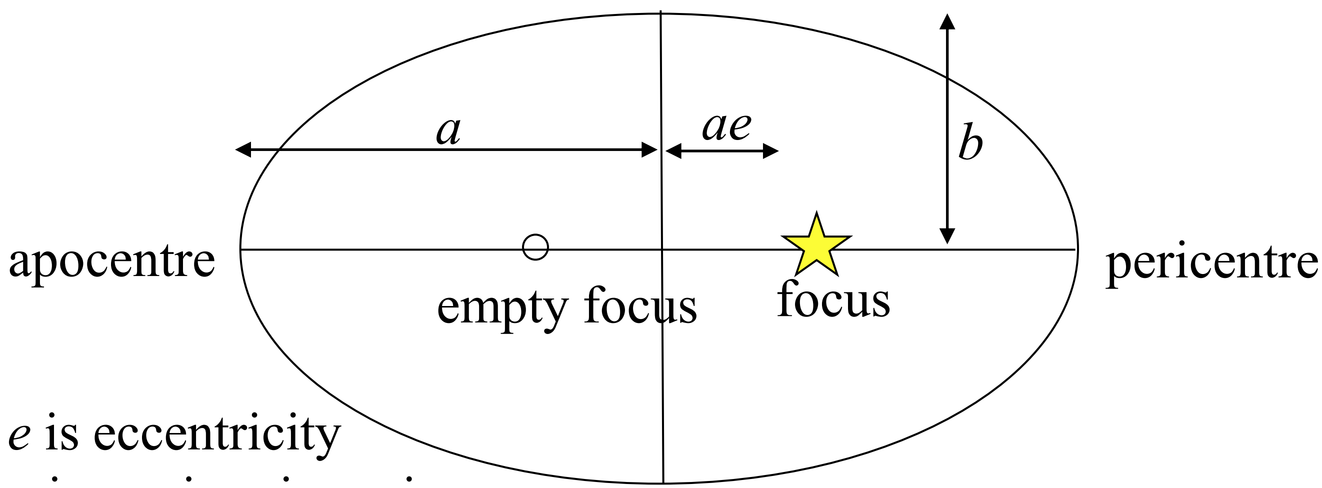
$$\frac{dE}{dt} = -\frac{63}{4\tilde{\mu}} \frac{e^2 n}{Q} \left(\frac{R}{a}\right)^5 \frac{Gm^2}{a}$$

Planning Ahead . . .

- Week 9
  - Tues 25th – Tides pt II
  - Thurs 27th – Case study I
- Week 10
  - Tues 1st – Case study II
  - Thurs 3rd – Revision lecture
- Final Exam – Mon 7th June 4:00-7:00 p.m.

Kepler's laws (1619)

- These were derived by *observation* (mainly thanks to Tycho Brahe – pre-telescope)
- 1) Planets move in ellipses with the Sun at one focus
- 2) A radius vector from the Sun sweeps out equal areas in equal time
- 3) (Period)^2 is proportional to (semi-major axis a)^3



e is eccentricity
a is semi-major axis

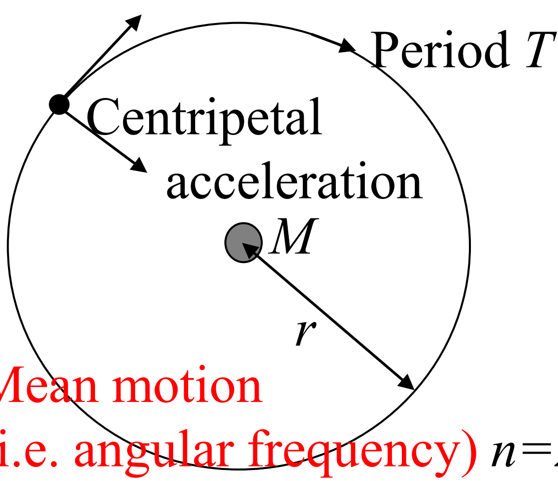
Newton (1687)

- Explained Kepler's observations by assuming an inverse square law for gravitation:

$$F = \frac{G m_1 m_2}{r^2}$$

Here F is the force acting in a straight line joining masses m_1 and m_2 separated by a distance r; G is a constant ($6.67 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$)

- A circular orbit provides a simple example and is useful for back-of-the-envelope calculations:



Centripetal acceleration = rn^2

Gravitational acceleration = GM/r^2

So $\mathbf{GM=r^3n^2}$ (also true for elliptical orbits)

So (period)^2 is proportional to r^3 (Kepler)

Mean motion (i.e. angular frequency) $n=2\pi/T$

Orbital angular momentum

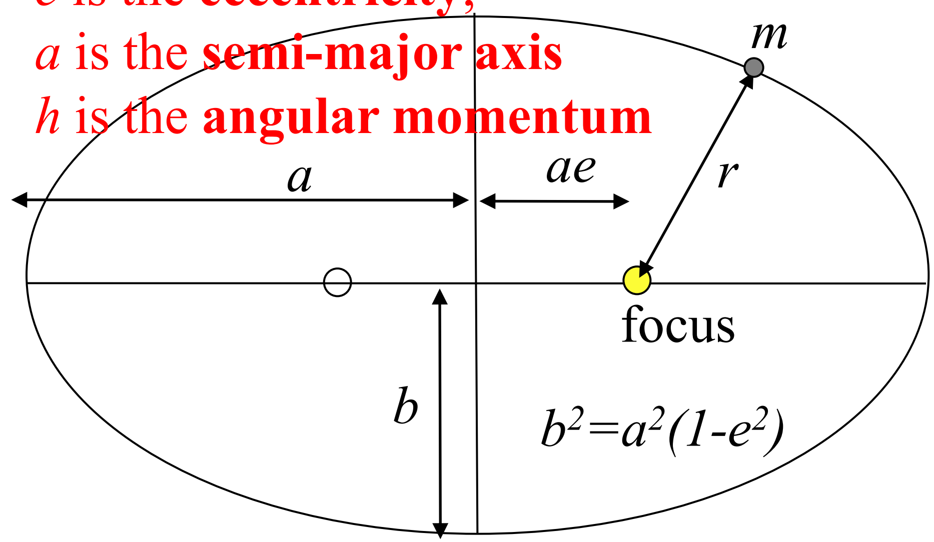
For a circular orbit:

Angular momentum = In

For a point mass, $I=ma^2$

Angular momentum/mass = na^2

e is the **eccentricity**,
a is the **semi-major axis**
h is the **angular momentum**



$$h = na^2\sqrt{1-e^2}$$

Angular momentum per unit mass.
Compare with na^2 for a circular orbit

An elliptical orbit has a *smaller* angular momentum than a circular orbit with the same value of a

Orbital angular momentum is *conserved* unless an external torque is acting upon the body

Energy

- To avoid yet more algebra, we'll do this one for circular coordinates. The results are the same for ellipses.
- Gravitational energy per unit mass

$$E_g=-GM/r$$

why the minus sign?

- Kinetic energy per unit mass

$$E_v=v^2/2=r^2n^2/2=GM/2r$$

- Total sum $E_g+E_v=-GM/2r$ (for elliptical orbits, $-GM/2a$)
- Energy gets exchanged between k.e. and g.e. during the orbit as the satellite speeds up and slows down
- But the *total* energy is constant, and independent of eccentricity
- Energy of rotation (spin) of a planet is

$$E_r=C\Omega^2/2$$

C is moment of inertia, Ω angular frequency

- Energy can be exchanged between orbit and spin, like momentum

Summary

- Mean motion of planet is independent of *e*, depends on *GM* and *a:*

$$n^2 a^3 = GM$$

- Angular momentum per unit mass of orbit is constant, depends on both *e* and *a:*

$$h = na^2\sqrt{1-e^2}$$

- Energy per unit mass of orbit is constant, depends only on *a:*

$$E = -\frac{GM}{2a}$$

Angular Momentum Conservation

- Angular momentum per unit mass

$$h = na^2\sqrt{1-e^2} = (GMa)^{1/2}\sqrt{1-e^2}$$

where the second term uses $n^2a^3 = GM$

- Say we have a primary with zero dissipation (this is *not* the case for the Earth-Moon system) and a satellite in an eccentric orbit.
- The satellite will still experience dissipation (because e is non-zero) – where does the energy come from?
- So a must decrease, but the primary is not exerting a torque; to conserve angular momentum, e must decrease also- circularization
- For small e, a small change in a requires a big change in e
- Orbital energy is *not* conserved – dissipation in satellite
- NB If dissipation *in the primary* dominates, the primary exerts a torque, resulting in angular momentum transfer from the primary's rotation to the satellite's orbit – the satellite (generally) moves out (as is the case with the Moon).

Summary

- Tidal bulge amplitude depends on mass, position, rigidity of body, and whether it is in synchronous orbit
- Tidal Love number is a measure of the amplitude of the tidal bulge compared to that of a uniform fluid body
- Tidal torques are responsible for orbital evolution e.g. orbit circularization, Moon moving away from Earth etc.
- Tidal strains cause dissipation and heating
- Orbits are described by mean motion n, semi-major axis a and eccentricity e.
- Orbital angular momentum is conserved in the absence of external torques: if a decreases, so does e