EART162: PLANETARY INTERIORS

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Last Week

• Elasticity: \( \sigma_{xx} = E \varepsilon_{xx} \)
  \( \sigma_{xy} = 2G \varepsilon_{xy} \)
  \( G = \frac{E}{2(1+v)} \)
  \( K = \rho \frac{dP}{d\rho} = \frac{E}{3(1-2v)} \)

• Flexural equation gives deflection \( w \) in response to load
  \( D \frac{d^4w}{dx^4} + (\rho_m - \rho_w)gw = q(x) \)

• The flexural parameter \( \alpha \) gives us the characteristic wavelength of deformation
  \( \alpha = \left[ \frac{4D}{g(\rho_m - \rho_w)} \right]^{1/4} \)
This Week – Heat Transfer

- See Turcotte and Schubert ch. 4
- Conduction, convection, radiation
- Radiation only important at or above the surface – not dealt with here
- Convection involves fluid motions – dealt with later in the course
- Conduction is this week’s subject

- Next week - Midterm
Conduction - Fourier’s Law

- Heat flow $F = k \frac{(T_1 - T_0)}{d} = k \frac{dT}{dz}$

- Heat flows from hot to cold (thermodynamics) and is proportional to the temperature gradient.

- Here $k$ is the thermal conductivity (Wm$^{-1}$K$^{-1}$) and units of $F$ are Wm$^{-2}$ (heat flux is per unit area).

- Typical values for $k$ are 2-4 Wm$^{-1}$K$^{-1}$ (rock, ice) and 30-60 Wm$^{-1}$K$^{-1}$ (metal).

- Solar heat flux at 1 A.U. is 1300 Wm$^{-2}$.

- Mean subsurface heat flux on Earth is 80 mWm$^{-2}$.

- What controls the surface temperature of most planetary bodies?
**Diffusion Equation**

- The **specific heat capacity** $C_p$ is the change in temperature per unit mass for a given change in energy: $\Delta E = mC_p\Delta T$

- We can use Fourier’s law and the definition of $C_p$ to find how temperature changes with time:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial z^2} = \kappa \frac{\partial^2 T}{\partial z^2}$$

- Here $\kappa$ is the **thermal diffusivity** ($=k/\rho C_p$) and has units of $\text{m}^2\text{s}^{-1}$.

- Typical values for rock/ice $10^{-6} \text{ m}^2\text{s}^{-1}$
Diffusion lengthscale (1)

• How long does it take a change in temperature to propagate a given distance?
• Consider an isothermal body suddenly cooled at the top
• The temperature change will propagate downwards a distance $d$ in time $t$

![Diagram showing temperature profile and depth]

• After time $t$, $F \sim k(T_1 - T_0)/d$
• The cooling of the near surface layer involves an energy change per unit area $\Delta E \sim d(T_1 - T_0)C_p\rho/2$
• We also have $Ft \sim \Delta E$
• This gives us $d^2 \sim k t$
Diffusion lengthscale (2)

• This is perhaps the single most important equation in the entire course
  \[ d^2 \sim \kappa t \]

• Another way of deducing this equation is just by inspection of the diffusion equation

• Examples:
  - 1. How long does it take to boil an egg?
    \( d \sim 0.02 \text{m}, \kappa = 10^{-6} \text{ m}^2\text{s}^{-1} \) so \( t \sim 6 \) minutes
  
  - 2. How long does it take for the molten Moon to cool?
    \( d \sim 1800 \text{ km}, \kappa = 10^{-6} \text{ m}^2\text{s}^{-1} \) so \( t \sim 100 \) Gyr.
    What might be wrong with this answer?
Heat Generation in Planets

• Most bodies start out hot (because of gravitational energy released during accretion)
• But there are also internal sources of heat
• For silicate planets, the principle heat source is radioactive decay (K,U,Th at present day)
• For some bodies (e.g. Io, Europa) the principle heat source is tidal deformation (friction)
• Radioactive heat production declines with time
• Present-day terrestrial value $\sim 5 \times 10^{-12}$ W kg$^{-1}$ (or $\sim 1.5 \times 10^{-8}$ W m$^{-3}$)
• Radioactive decay accounts for only about half of the Earth’s present-day heat loss (why?)
Example - Earth

- Near-surface consists of a mechanical boundary layer (plate) which is too cold to flow significantly (Lecture 3)
- The base of the m.b.l. is defined by an isotherm (~1400 K)
- Heat must be transported across the m.b.l. by conduction
- Let’s assume that the heat transported across the m.b.l. is provided by radioactive decay in the mantle (true?)

By balancing these heat flows, we get

$$d = \frac{3k\Delta T}{HR}$$

Here $H$ is heat production per unit volume, $R$ is planetary radius.

Plugging in reasonable values, we get m.b.l. thickness $d=225$ km and a heat flux of 16 mWm$^{-2}$. Is this OK?
Internal Heat Generation

• Assume we have internal heating $H$ (in Wkg$^{-1}$)
• From the definition of $C_p$ we have $Ht = \Delta TC_p$
• So we need an extra term in the heat flow equation:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} + \frac{H}{C_p}$$

• This is the one-dimensional, Cartesian thermal diffusion equation assuming no motion
• In steady state, the LHS is zero and then we just have heat production being balanced by heat conduction
• The general solution to this steady-state problem is:

$$T = a + bz - \frac{H}{2\kappa C_p} z^2$$
Example

• Let’s take a spherical, conductive planet in steady state

• In spherical coordinates, the diffusion equation is:

\[
\frac{\partial T}{\partial t} = \kappa \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{H}{C_p} = 0
\]

• The solution to this equation is

\[
T(r) = T_s + \frac{\rho H}{6k} (R^2 - r^2)
\]

Here \(T_s\) is the surface temperature, \(R\) is the planetary radius, \(\rho\) is the density

• So the central temperature is \(T_s + (\rho HR^2/6k)\)

• E.g. Earth \(R=6400\) km, \(\rho=5500\) kg m\(^{-3}\), \(k=3\) W m\(^{-1}\) K\(^{-1}\), \(H=6\times10^{-12}\) W kg\(^{-1}\) gives a central temp. of \(\sim75,000\)K!

• What is wrong with this approach?
What happens if the medium is moving?

- Two ways of looking at the problem:
  - Following an individual particle – **Lagrangian**
  - In the laboratory frame - **Eulerian**

  ![Diagram of Particle Frame and Laboratory Frame with Temperature Contours]

  - In the particle frame, there is no change in temperature with time.
  - In the laboratory frame, the temperature at a fixed point is changing with time.
  - But there would be no change if the temperature gradient was perpendicular to the velocity.

In the Eulerian frame, we write

\[
\frac{\partial T}{\partial t} = -u \cdot \nabla T
\]

Where does this come from? What does \( \nabla T \) mean?

The right-hand side is known as the **advected** term.
Material Derivative

- So if the medium is moving, the heat flow equation is

\[ \frac{DT}{Dt} = \kappa \frac{\partial^2 T}{dz^2} + \frac{H}{C_p} \]

- Here we are using the material derivative \( D/Dt \), where

\[ \frac{D}{Dt} = \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \]

- It is really just a shorthand for including both the local rate of change, and the advective term
- It applies to the Eulerian (laboratory) reference frame
- Not just used in heat transfer (\( T \)). Also fluid flow (\( u \)), magnetic induction etc.
Skin-Depth Problem

• Let’s go back to the original diffusion equation:
  \[
  \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}
  \]

• Say we have a surface temperature which varies periodically (e.g. day-night, yearly etc.) with frequency \( \omega \)

• How deep do these temperature changes penetrate?

• The full solution is annoying and involves separation of variables (see T&S Section 4-14)

• But there’s a quick way to solve this problem using \( d^2 \sim \kappa t \)

• This approach gives us a **skin depth** of \( d \sim (2\pi \kappa/\omega)^{1/2} \) which is very close to the full solution of \( d = (2\kappa/\omega)^{1/2} \)

  Note that \( \omega = 2\pi/\text{period!} \)
Skin-Depth and Thermal Inertia

• Temperature fluctuations are damped at depth; higher frequency fluctuations are damped at shallower depths
• The skin depth tells us how thick a layer feels the effect of the changing surface temperature
• On a planetary surface, the power input (radiation) varies periodically e.g. $F=F_0 \sin(\omega t)$
• The resulting change in the temperature of the near-surface layer is given by:
  \[ \Delta T \sim \frac{F_0 t^{1/2}}{(k \rho C_p)^{1/2}} = \frac{F_0 t^{1/2}}{I} \]
• The quantity $I$ is the thermal inertia, which tells us how rapidly the temperature of the surface will change
Why is thermal inertia useful?

- The main controls on thermal inertia are the physical properties of the near-surface materials e.g. particle size and rock vs. sand fraction (rocks have a higher $I$ and thus take longer to heat up or cool down)
- Thermal inertia (and thus these physical properties) can be measured remotely – infra-red cameras on spacecraft can track the changing temperature of the surface as a function of time

University of Colorado
Map of thermal inertia of Mars (Mellon et al. 2000)

Low $I$ – dust bowl
High $I$ – lava flows
Thermal Stresses

• Recall thermal expansivity $\alpha$: materials expand if heated and cool if contracted ($\alpha \sim 10^{-5}$ K$^{-1}$ for rock)

$$\varepsilon = -\alpha \Delta T$$  (Contraction strain is negative)

• Say we have plane strain, confined so that $\varepsilon_1 = \varepsilon_2 = 0$

$$\varepsilon_1 = \frac{1}{E} (\sigma_1 - \nu \sigma_2) - \alpha \Delta T$$

$$\varepsilon_2 = \frac{1}{E} (\sigma_2 - \nu \sigma_1) - \alpha \Delta T$$  (See Week 3)

• This results in $\sigma_1 = \sigma_2 = \frac{E \alpha \Delta T}{1 - \nu}$

• E.g. during an ice age $\Delta T = 10$ K, $\sigma = 10$ MPa i.e. a lot!

• Other applications: cooling lithospheres, rotating bodies, badly-insulated spacecraft . . .
Deformation Heating

- Energy per unit volume $W$ required to cause a given amount of strain $\varepsilon$:
  $$ W = \frac{1}{2} E \varepsilon^2 $$

- Power $P$ per unit volume is:
  $$ P = E \varepsilon \dot{\varepsilon} = \sigma \dot{\varepsilon} $$

- So power depends on stress and strain rate.

- E.g. long-term fault slip $\dot{\varepsilon} = 10^{-15}$ s$^{-1}$, $\sigma = 10$ MPa, $P = 10^{-8}$ Wm$^{-3}$ – comparable to mantle heat production.

- A particularly important sort of deformation heating is that due to solid body tides.

*Diurnal* tides can be large e.g. 30m on Europa.

*Eccentric* orbit.

Eccentric orbit.

Satellite

Jupiter
Tidal Heating (1)

• A full treatment is beyond the scope of this course, but here’s an outline

• Strain depends on tidal amplitude $H$

\[ \varepsilon \approx \frac{H}{R} \]

• Strain rate depends on orbital period $\tau$

• What controls the tidal amplitude?

• Combining the various pieces, we get

\[ P \approx \frac{EH^2}{QR^2\tau} \]

• Here $Q$ is a dimensionless factor telling us what fraction of the elastic energy is dissipated each cycle

• Example: Io $H=300$ m, $Q=100$, $R=1800$ km, $\tau=1.8$ days, $E=10$ GPa (why?). This gives us $P=2\times10^{-5}$ Wm$^{-3}$
Tidal Heating (2)

• $P=2\times10^{-5}$ Wm$^{-3}$ results in a surface heat flux of 12W m$^{-2}$ (about as much energy as Io receives from the Sun!)

• Is this a reasonable estimate?

• Tides can be the dominant source of energy for satellites orbiting close to giant planets.
Summary

• Everything you need to know about heat conduction in one equation: $d^2 = \kappa t$

• Heat transport across mechanical boundary layer is usually by conduction alone

• Heat is often transported within planetary interiors by convection (next lecture)

• Main source of heat in silicate planets is radioactive decay

• Tidal heating can be an important source of heat in bodies orbiting giant planets
Supplementary Material Follows
Conductive half-space cooling problem

- We are interested in how rapidly a temperature change propagates
- We have already found a scaling argument: \( t \sim d^2/k \)
- Now we’ll take a more rigorous approach (see T&S 4-15)

Governing equation:

\[
\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{dz^2}
\]

To simplify, we non-dimensionalize the temperature:

\[
\theta = \frac{T - T_s}{T_m - T_s}
\]

The governing equation doesn’t change, but the boundary conditions become simpler:

\( \theta(z,0)=1, \theta(0,t)=0, \theta(inf,t)=1 \)
Conductive half-space (cont’d)

\[ \frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2} \]

- The problem has only one length-scale: \((\kappa t)^{1/2}\)
- We can go from 2 variables \((t,z)\) to one by employing a similarity variable \(\eta:\)
  \[ \eta = \frac{z}{2\sqrt{\kappa t}} \]

- This approach assumes that solutions at different times will look the same if the lengths are scaled correctly
- So we can rewrite the original diffusion equation:

\[ -\eta \frac{d \theta}{d \eta} = \frac{1}{2} \frac{d^2 \theta}{d \eta^2} \quad \theta(\text{inf}) = 1, \ \theta(0) = 0 \]
Conductive half-space (cont’d)

\[-\eta \frac{d \theta}{d \eta} = \frac{1}{2} \frac{d^2 \theta}{d \eta^2}\]

\[\theta(\text{inf}) = 1, \ \theta(0) = 0\]

Solution: \[\theta = \text{erf}(\eta)\]

Where \(\text{erf}\) is the error function:

\[\text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta'^2} \, d\eta'\]

Re-dimensionalize:

\[\frac{T - T_s}{T_m - T_s} = \text{erf} \frac{z}{2\sqrt{\kappa t}}\]

So e.g. for any combination such that \(z/2(\kappa t)^{1/2} = 1\), we have \(T - T_s = 0.15(T_m - T_s)\)
So what?

• Let’s say the characteristic cooling timescale $t_c$ is the time for the initial temperature contrast to drop by 85%

• From the figure on the previous page, we get

$$t_c = \frac{z^2}{4K}$$

• This should look *very* familiar (give or take a factor of 4)

• So the sophisticated approach gives an almost identical result to the one-line approach

• An *identical* equation arises when we consider the solidification of material (the Stefan problem), except that the 4 is replaced by a factor of similar size which depends on latent heat and heat capacity (see T&S 4-17)
Earth (cont’d)

• Predicted m.b.l. thickness 90 km
• This underestimates continental m.b.l. thickness by a factor of ~2
• The main reason is that (oceanic) plates are thinner near spreading centres, and remove more heat than the continents
• Our technique would work better on planets without plate tectonics

This figure shows continental geotherms based on P,T data from nodules. The geotherm is clearly conductive. Note the influence of the crust. The mantle heat flux is lower than our estimate on the previous slide, and the m.b.l. thicker.
Peclet Number

• It would be nice to know whether we have to worry about the advection of heat in a particular problem

• One way of doing this is to compare the relative timescales of heat transport by conduction and advection:

\[ t_{\text{cond}} \sim \frac{L^2}{\kappa} \quad t_{\text{adv}} \sim \frac{L}{u} \quad Pe \sim \frac{uL}{\kappa} \]

• The ratio of these two timescales is called a dimensionless number called the Peclet number \( Pe \) and tells us whether advection is important

• High \( Pe \) means advection dominates diffusion, and \( \text{v.v.} \) *

• E.g. lava flow, \( u \sim 1 \text{ m/s}, L \sim 10 \text{ m}, Pe \sim 10^7 \) : advection is important

* Often we can’t ignore diffusion even for large \( Pe \) due to stagnant boundary layers
Viscous Heating

- Power = force x velocity
- Viscous flow involves shear stresses – potential source of power (heat)

- We can find the power dissipated per unit volume $P$

$$ P = \mu \left( \frac{\partial u}{\partial z} \right)^2 = \mu \dot{\varepsilon}^2 $$

- Note the close resemblance to the equation on the earlier slide (substitute $\mu = \sigma / \dot{\varepsilon}$)

- Is viscous heating important in the Earth’s mantle?

- Can you think of a situation in which it might be important?