EART162: PLANETARY INTERIORS

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Last Week

- Elasticity: Young’s modulus = stress / strain (also Poisson’s ratio – *what does it do?*)
- Another important (*why?*) variable is the bulk modulus, which tells us how much pressure is required to cause a given change in density. *Definition?*
- Flow law describes the relationship between stress and strain rate for geological materials
- (Effective) viscosity is stress / strain rate
- Viscosity is very temperature-dependent: \(\exp(-Q/RT)\)
This Week – Isostasy and Flexure

• See Turcotte and Schubert chapter 3
• How are loads supported?
• Isostasy – zero elastic strength
• Flexure – elastic strength (rigidity) is important
• What controls rigidity?
• We can measure rigidity remotely, and it tells us about a planet’s thermal structure
Airy Isostasy

- Let’s start by assuming that the crust and mantle are unable to support loads elastically

\[
\begin{align*}
\text{Crust } & \rho_c \\
\text{Mantle } & \rho_m
\end{align*}
\]

- The crust will deflect downwards until the surface load (mass excess) is balanced by a subsurface “root” (mass deficit – dense mantle replaced by light crust)

- 90% of an iceberg is beneath the surface of the ocean for exactly the same reason

- This situation is called (Airy) isostasy
Consequences of Isostasy

- In the case of no elastic strength, the load is balanced by the mantle root: $h \rho_c = r(\rho_m - \rho_c)$

- This also means that there are no lateral variations in pressure beneath the crustal root.

- So crustal thickness contrasts ($\Delta t_c = h + r$) lead to elevation contrasts ($h$):
  $$h = \frac{\rho_m - \rho_c}{\rho_m} \Delta t_c$$

- Note that the elevation is independent of the background crustal thickness $t_c$. 

\[ \text{Crust } \rho_c \quad \text{Mantle } \rho_m \]
Pratt Isostasy

- Similar principle – pressures below some depth do not vary laterally
- Here we do it due to variations in density, rather than crustal thickness

\[ h = \frac{(\rho_2 - \rho_1)}{\rho_2} t_c \]

What’s an example of where this mechanism occurs on Earth?
Gravity Effects

• Because there are no lateral variations in pressure beneath a certain depth, that means that the total mass above this depth does not vary laterally either.

• So what sort of gravity anomalies should we see?

• Very small ones!

(NB there will actually be a small gravity anomaly and edge effects in this case)

So we can use the size of the gravity anomalies to tell whether or not surface loads are compensated.
Example - Mars

• The southern half of Mars is about 3 km higher than the northern half (why?)

• But there is almost no gravity anomaly associated with this “hemispheric dichotomy”

• We conclude that the crust of Mars here must be compensated (i.e. weak)

• Pratt isostasy? Say $\rho_1=2700 \text{ kgm}^{-3}$ (granite) and $\rho_2=2900 \text{ kgm}^{-3}$ (basalt). This gives us a crustal thickness of 45 km
Mars (cont’d)

- On the other hand, some of the big volcanoes (24 km high) have gravity anomalies of 2000-3000 mGal
- If the volcanoes were sitting on a completely rigid plate, we would expect a gravity anomaly of say $2.9 \times 24 \times 42 \approx 2900$ mGal
- We conclude that the Martian volcanoes are almost uncompensated, so the crust here is very rigid
- Remember that what’s important is the strength of the crust at the time the load was emplaced – this may explain why different areas have different strengths
Flexure

- So far we have dealt with two end-member cases: when the lithosphere is completely rigid, and when it has no strength at all (isostasy)
- It would obviously be nice to be able to deal with intermediate cases, when a load is only partly supported by the rigidity of the lithosphere
- I’m not going to derive the key equation – see the Supplementary Section (and T&S Section 3-9) for the gory details
- We will see that surface observations of deformation can be used to infer the rigidity of the lithosphere
- Measuring the rigidity is useful because it is controlled by the thermal structure of the subsurface
In general, a load will be supported by a combination of elastic stresses and buoyancy forces (due to the different density of crust and mantle).

The elastic stresses will be both compressional and extensional (see diagram).

Note that in this example the elastic portion includes both crust and mantle.
Flexural Equation (1)

\[ D \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} + (\rho_m - \rho_w)gw = q(x) \]

- \( D \) is the (flexural) rigidity, \( T_e \) is the elastic thickness

\[ D = \frac{ET_e^3}{12(1 - v^2)} \]
Flexural Equation (2)

- Here the load \( q(x) = \rho_l gh \)
- We’ll also set \( P=0 \)
- The flexural equation is:

\[
D \frac{d^4 w}{dx^4} + (\rho_m - \rho_l)gw = \rho_l gh
\]

- If the plate has no rigidity, \( D=0 \) and we get

\[
(\rho_m - \rho_l)w = h\rho_l
\]

- This is just the expression for **Airy isostasy**
- So if the flexural rigidity is zero, we get isostasy
Flexural Equation (3)

\[ D \frac{d^4 w}{dx^4} + (\rho_m - \rho_t)gw = \rho_t gh \]

- Let's assume a sinusoidal variation in loading \( h = h_0 e^{ikx} \)
- Then the response must also be sinusoidal \( w = w_0 e^{ikx} \)
- We can relate \( h_0 \) to \( w_0 \) as follows

\[ w_0 = \frac{\rho_t}{\Delta \rho + \frac{Dk^4}{g}} h_0 \]

Here \( \Delta \rho = \rho_m - \rho_t \) and \( k = 2\pi/\lambda \), where \( \lambda \) is the wavelength

- Does this expression make sense?
- What happens if \( D = 0 \) or \( \Delta \rho = 0 \)?
- What happens at very short or very long wavelengths?
Degree of Compensation

- The deflection caused by a given load: \( w_0 = \frac{\rho_l}{\Delta \rho + \frac{Dk^4}{g}} h_0 \)
- We also know the deflection in the case of a completely compensated load \((D=0)\): \( w_1 = \frac{\rho_l}{\Delta \rho} h_0 \)
- The degree of compensation \( C \) is the ratio of the deflection to the deflection in the compensated case:
  \[ C = \frac{w_0}{w_1} = \frac{1}{1 + \frac{Dk^4}{g\Delta \rho}} \]
- Long \( \lambda \), \( C \sim 1 \) (compensated); short \( \lambda \), \( C \sim 0 \) (uncomp.)
- \( C \sim 1 \) gives small gravity anomalies, \( C \sim 0 \) large ones
- Critical wavenumber: \( C = 0.5 \) means \( k = (\Delta \rho g / D)^{1/4} \)
Example

- Let’s say the elastic thickness on Venus is 30 km (we’ll use $E=100$ GPa, $v=0.25$, $g=8.9$ ms$^{-2}$, $\Delta \rho=500$ kg m$^{-3}$)
- The rigidity $D=ETe^3/12(1-v^2) \sim 2 \times 10^{23}$ Nm
- The critical wavenumber $k=(\Delta \rho g / D)^{1/4} \sim 1.3 \times 10^{-5}$ m$^{-1}$
- So the critical wavelength $\lambda=2\pi/k=500$ km

Would we expect it to be compensated or not? What kind of gravity anomaly would we expect?
Degree of Compensation (2)

What gravity signals are associated with $C=1$ and $C=0$?

How would the curves move as $T_e$ changes?

So by measuring the ratio of gravity to topography (admittance) as a function of wavelength, we can infer the elastic thickness of the lithosphere remotely.
Flexural Parameter (1)

- Consider a line load acting on a plate:

- Except at \( x=0 \), load=0 so we can write:
  \[
  D \frac{d^4 w}{dx^4} + (\rho_m - \rho_w)gw = 0
  \]

- Boundary conditions for an unbroken plate are that \( dw/dx=0 \) at \( x=0 \) and \( w \to 0 \) as \( x \to \infty \)

- The solution is \( w = w_0 \exp(-x/\alpha)(\cos\frac{x}{\alpha} + \sin\frac{x}{\alpha}) \)

- Here \( \alpha \) is the flexural parameter

(Note the similarity of \( \alpha \) to the critical wavenumber)
Flexural Parameter (2)

- Flexural parameter $\alpha = (4D/g(\rho_m - \rho_w))^{1/4}$
- It is telling us the *natural wavelength* of the elastic plate
- E.g. if we apply a concentrated load, the wavelength of the deformation will be given by $\alpha$
- Large $D$ gives long-wavelength deformation, and v.v.
- If the load wavelength is $\gg \alpha$, then the deformation will approach the isostatic limit (i.e. $C \sim 1$)
- If the load wavelength is $\ll \alpha$, then the deformation will be small ($C \sim 0$) and have a wavelength given by $\alpha$
- If we can measure a flexural wavelength, that allows us to infer $\alpha$ and thus $D$ or $T_e$ directly. This is useful!
Example

- This is an example of a profile across a rift on Ganymede
- An eyeball estimate of $\alpha$ would be about 10 km
- For ice, we take $E=10$ GPa, $\Delta\rho=900$ kg m$^{-3}$ (there is no overlying ocean), $g=1.3$ ms$^{-2}$

- If $\alpha=10$ km then $D=2.9\times10^{18}$ Nm and $T_e=1.5$ km
- A numerical solution gives $T_e=1.4$ km – pretty good!
- So we can determine $T_e$ remotely
- This is useful because $T_e$ is ultimately controlled by the temperature structure of the subsurface
$T_e$ and temperature structure

- Cold materials behave elastically
- Warm materials flow in a viscous fashion
- This means there is a characteristic temperature (roughly 70% of the melting temperature) which defines the base of the elastic layer

- E.g. for ice the base of the elastic layer is at about 190 K
  - The measured elastic layer thickness is 1.4 km (from previous slide)
  - So the thermal gradient is 60 K/km
  - This tells us that the (conductive) ice shell thickness is 2.7 km (!)
$T_e$ and age

- The elastic thickness recorded is the lowest since the episode of deformation.
- In general, elastic thicknesses get larger with time (why?)
- So by looking at features of different ages, we can potentially measure how $T_e$, and thus the temperature structure, have varied over time.
- This is important for understanding planetary evolution.

McGovern et al., *JGR* 2002
$T_e$ in the solar system

- Remote sensing observations give us $T_e$
- $T_e$ depends on the composition of the material (e.g. ice, rock) and the temperature structure
- If we can measure $T_e$, we can determine the temperature structure (or heat flux)
- Typical (approx.) values for solar system objects:

<table>
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<th>Body</th>
<th>$T_e$ (km)</th>
<th>$dT/dz$ (K/km)</th>
<th>Body</th>
<th>$T_e$</th>
<th>$dT/dz$ (K/km)</th>
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<tbody>
<tr>
<td>Earth (cont.)</td>
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<td>15</td>
<td>Venus (450°C)</td>
<td>30</td>
<td>15</td>
</tr>
<tr>
<td>Mars (recent)</td>
<td>100</td>
<td>5</td>
<td>Moon (ancient)</td>
<td>15</td>
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</tr>
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<td>40</td>
<td>Ganymede</td>
<td>2</td>
<td>40</td>
</tr>
</tbody>
</table>
Summary

• **Flexural equation** determines how loads are supported:

\[ D \frac{d^4 w}{dx^4} + (\rho_m - \rho_w)gw = q(x) \]

• The **flexural parameter** \( \alpha \) gives us the characteristic wavelength of deformation

\[ \alpha = \left[ \frac{4D}{g(\rho_m - \rho_w)} \right]^{1/4} \]

• Loads with wavelengths >> \( \alpha \) are isostatically supported
• Loads with wavelengths << \( \alpha \) are elastically supported
• We can infer \( \alpha \) from looking at flexural topography (or by using gravity & topography together – admittance)
• Because the rigidity depends on the temperature structure, determining \( \alpha \) allows us to determine \( dT/dz \)
Supplementary Material follows
Plate Bending

- Bending an elastic plate produces both compressional and extensional strain.
- The amount of strain depends on the radius of curvature $R$.

$$\text{Strain} = \varepsilon_{xx} \frac{\Delta l}{l} = \frac{y\phi}{R\phi} = \frac{y}{R}$$

- Note that there is no strain along the centre line ($y=0$).
- The resulting stress is given by (see T&S):

$$\sigma_{xx} = \frac{E}{1-v^2} \varepsilon_{xx}$$
Radius of Curvature

What is the local radius of curvature of a deforming plate? Useful to know, because that allows us to calculate the local stresses.

It can be shown that

\[ \frac{1}{R} = -\frac{d^2w}{dx^2} \]

So the bending stresses are given by

\[ \sigma_{xx} = -y \frac{E}{1-v^2} \frac{d^2w}{dx^2} \]

(y is the distance from the mid-plane)
Bending Moment

- Balance torques: \( V\, dx = dM \)
- Balance forces: \( q\, dx + dV = 0 \)
- Put the two together:
  \[
  \frac{d^2 M}{dx^2} = -q \tag{B}
  \]

Does this make sense?

Moment:
\[
M = \int_{-T_e/2}^{T_e/2} \sigma_{xx} \, y \, dy
\]
Putting it all together . . .

• Putting together A, B and C we end up with

$$\frac{ET_e^3}{12(1-\nu^2)} \frac{d^4w}{dx^4} = D \frac{d^4w}{dx^4} = q(x)$$

• Here $D$ is the rigidity

• Does this equation make sense?
Admittance Example

- Comparison of Hawaii and Ulfrun Regio (Venus)
- What is happening on Venus at short wavelengths?
- Are you surprised that the two elastic thicknesses are comparable?

Nimmo and McKenzie 1998