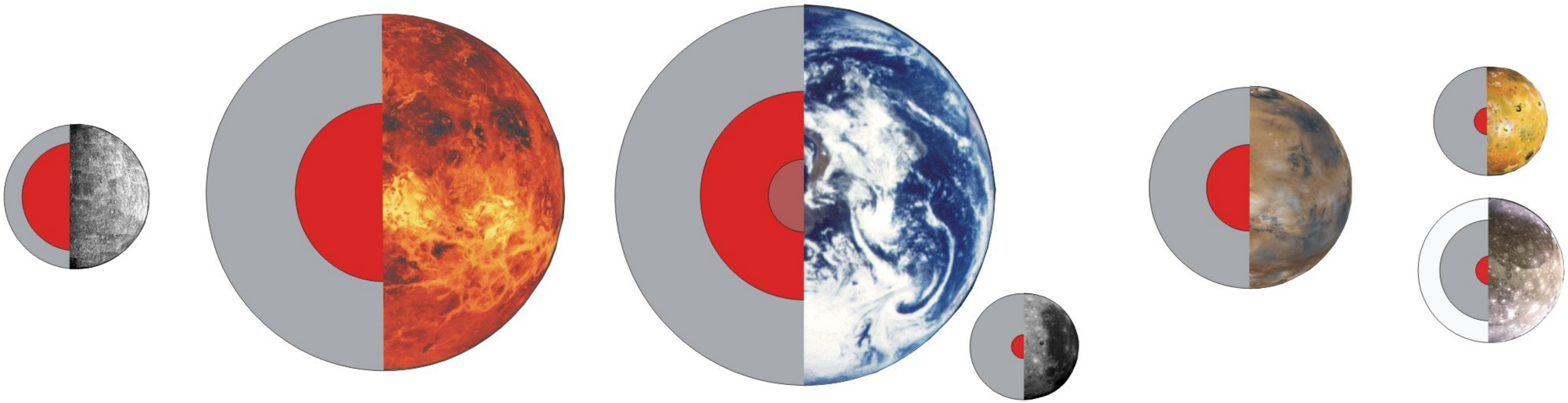
EART162: PLANETARY INTERIORS



Francis Nimmo

Last Week

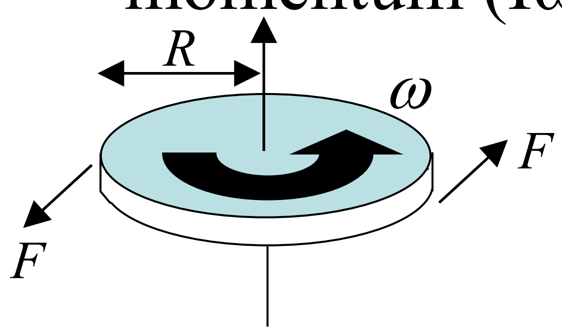
- Solar system formation
- Composition of solar nebular
  - Solar photosphere
  - Carbonaceous chondrites
- Samples of planetary interiors (Moon, Earth, Mars, Vesta)
- Bulk density inferred from gravity
- Accretionary processes
  - Gravitational energy considerations
  - Consequences: heating and differentiation
- Building a generic terrestrial planet

This Week – Moment of Inertia

- Gravity gives us the mass/density of a planet. How?
- Why is this useful? Density provides constraints on interior structure
- We can obtain further constraints on the interior structure from the *moment of inertia*
  - How do we obtain it?
  - What does it tell us?
- We can also use gravity to investigate lateral variations in the subsurface density
- See Turcotte and Schubert chapter 5

Moment of Inertia (1)

- The moment of inertia (MoI) is a measure of an object's resistance to being "spun up" or "spun down"
- In many ways analogous to mass, but for rotation
- MoI must always be measured about a particular axis (the axis of rotation)
- The MoI is governed by the *distribution of mass* about this axis (mass further away = larger MoI)
- Often abbreviated as *I*; also *A,B,C* for planets
- In the absence of external forces (torques), angular momentum (Iω) is *conserved* (ice-skater example)



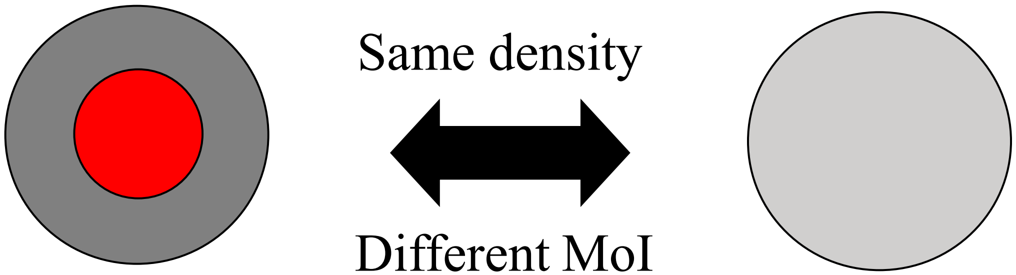
Linear acceleration: $F = m\frac{dv}{dt}$

Rotational acceleration: $\boxed{T = I\frac{d\omega}{dt}}$

(*T* is torque (=2 *F R*))

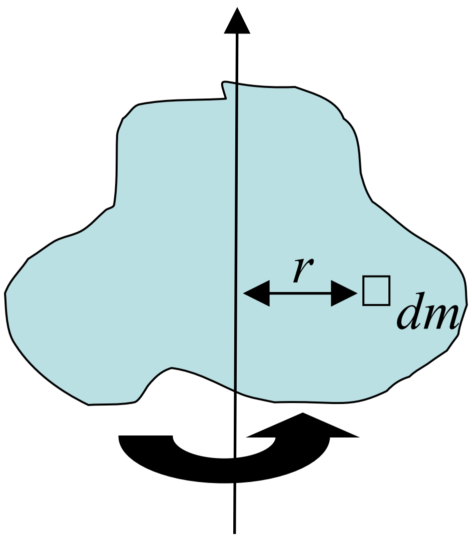
Moment of Inertia (2)

- MoI is useful because we can measure it remotely, and it tells us about *distribution of mass* (around an axis)
- This gives us more information than density alone



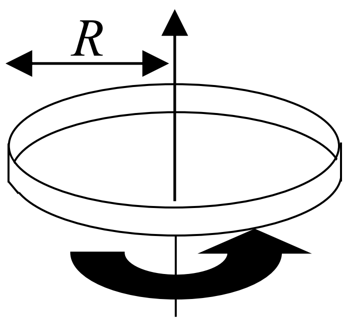
- Calculating MoI is straightforward (in theory):

$$I = \sum mr^2 = \int r^2 dm$$



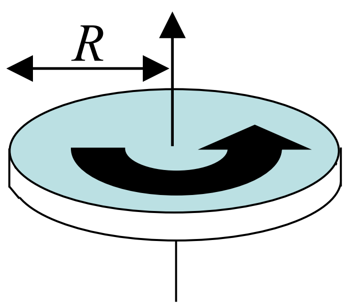
Calculating MoI

- Some simple examples (before we get to planets)

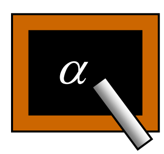


Uniform hoop – by inspection

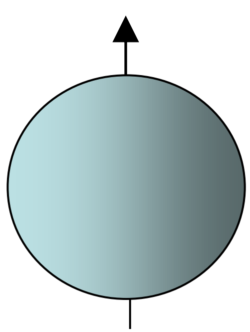
$$I = MR^2$$



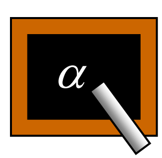
Uniform disk – requires integration



$$I = 0.5\ MR^2$$



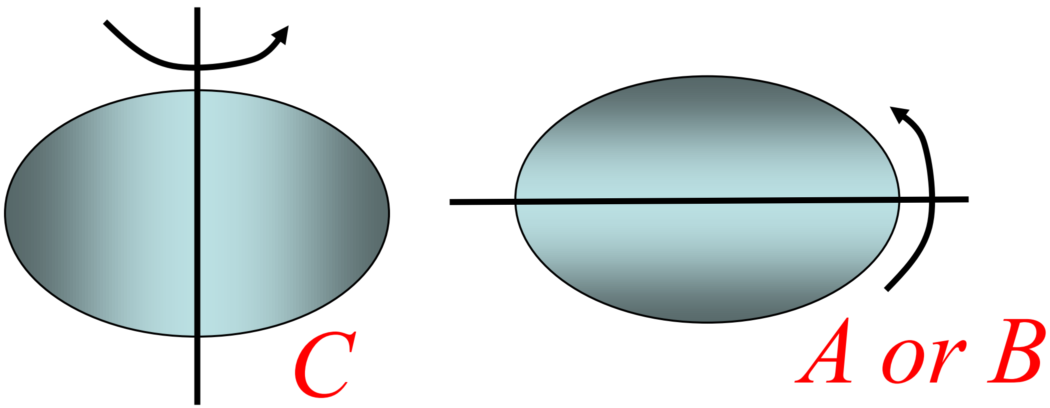
Uniform sphere – this is one to remember because it is a useful comparison to real planets



$$\boxed{I = 0.4\ MR^2}$$

Moments of inertia of a planet

- Planets are *flattened* (because of rotation - centripetal)
- This means that their moments of inertia (*A*,*B*,*C*) are different. By convention *C*>*B*>*A*
- *C* is the moment *about the axis of rotation*

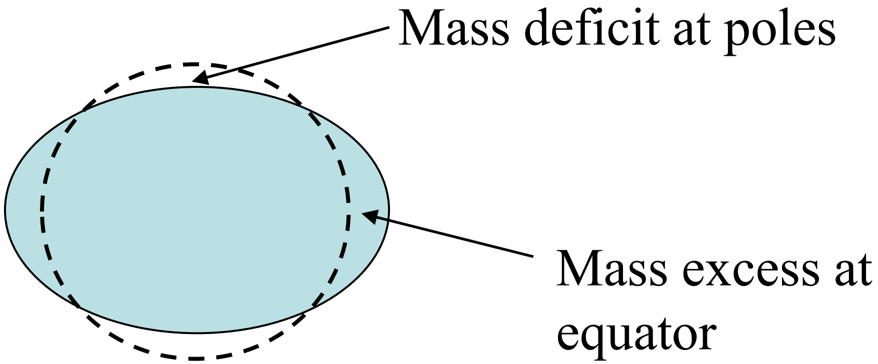


In general, A and B are approximately equal

- The *difference* in moments of inertia (*C*-*A*) is an indication of how much excess mass is concentrated towards the equator

Moment of Inertia Difference

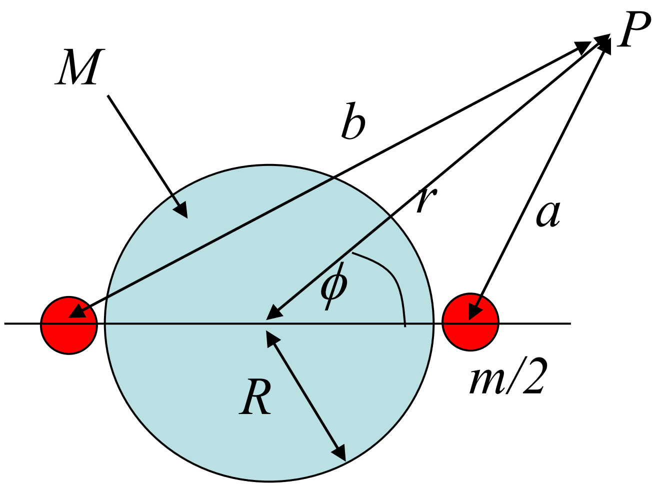
- Because a moment of inertia difference indicates an *excess in mass* at the equator, there will also be a corresponding effect on the gravity field
- So we can use observations of the gravity field to infer the moment of inertia difference
- The effect on the gravity field will be a function of position (+ at equator, - at poles)



How do we use the gravity to infer the moment of inertia difference?

Relating *C-A* to gravity (1)

- Here is a simple example which gives a result comparable to the full solution
- See T&S Section 5.2 for the full solution (tedious)



We represent the equatorial bulge as two extra blobs of material, each of mass *m/2*, added to a body of mass *M*.
We can calculate the resulting MoI difference and effect on the gravitational acceleration as a function of latitude ϕ.

Point source: $\dfrac{G(M+m)}{r^2}$ Extra term from bulge: $-\dfrac{3GmR^2}{r^4}\left[\frac{4}{3}\sin^2\phi - 1\right]$

Corresponding increase in C : mR^2 increase in *A* or *B*: 0

So now we have a description of the gravity field of a flattened body, and its MoI difference ($C\text{-}A = mR^2$)

Gravity field of a flattened planet

- The full solution is called MacCullagh's formula:

$$g = \frac{GM}{r^2} - \frac{3G\boxed{(C-A)}}{2r^4}\left[3\sin^2\phi - 1\right]$$

Point source Contribution from bulge MoI difference

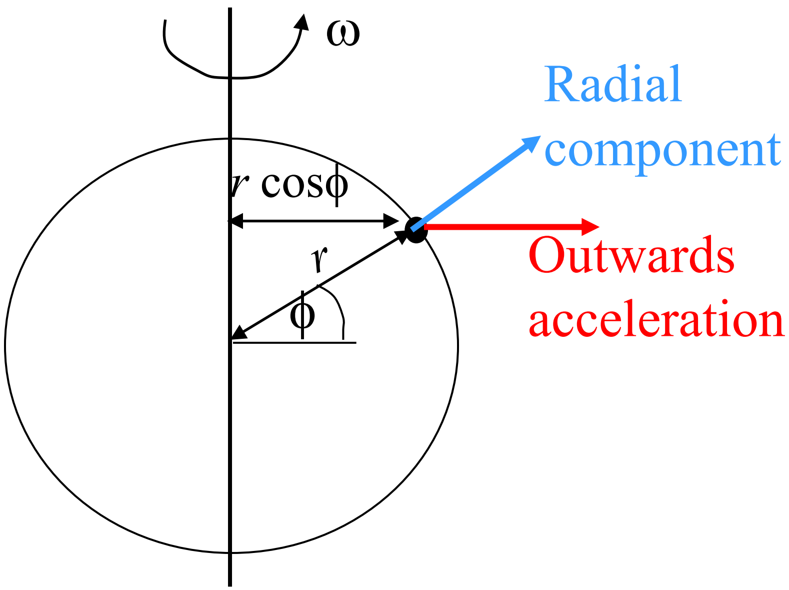
- Note the similarities to the simplified form derived on the previous page
- So we can use a satellite to measure the gravity field as a function of distance r and latitude ϕ, and obtain C-A
- We'll discuss how to get C from C-A in a while
- The MoI difference is often described by J_2, where

$$J_2 = \frac{C-A}{Ma^2}$$

(J_2 is dimensionless, a is the equatorial radius)

Effect of rotation

- Final complication – a body on the surface of the planet experiences rotation and thus a *centripetal acceleration*
- Effect is pretty straightforward:



Radial component of acceleration:

$$-\omega^2 r \cos^2\phi$$

where ω is the angular velocity

So the complete formula for acceleration *g* on a planet is:

$$g = \frac{GM}{r^2} - \frac{3GMa^2J_2}{2r^4}[3\sin^2\phi - 1] - \omega^2 r \cos^2\phi$$

Gravitational Potential

- Gravitational potential is the work done to bring a unit mass from infinity to the point in question:

$$U = \int_{\infty}^{r} \frac{F(r)}{m} dr = \int_{\infty}^{r} g(r) dr$$

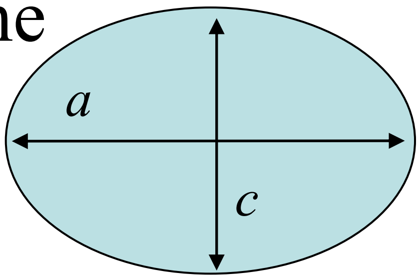
- For a spherically symmetric body, *U=-GM/r*
- Why is this useful?
- For a rotationally flattened planet, we end up with:

$$U = -\frac{GM}{r} + \frac{GMa^2 J_2}{2r^3}[3\sin^2\phi - 1] - \tfrac{1}{2}\omega^2 r^2 \cos^2\phi$$

- This is useful because a fluid will have the *same potential* everywhere on its surface – so we can predict the shape of a rotating fluid body

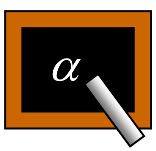
Rotating Fluid Body Shape

- For a fluid, the grav. potential is the same everywhere on the surface
- Let's equate the polar and equatorial potentials for our rotating shape, and let us also define the ellipticity (or flattening): $f = \dfrac{a-c}{a}$



- After a bit of algebra, we end up with:

Note approximate!



$$f \approx \frac{3}{2} J_2 + \frac{1}{2} \frac{a^3 \omega^2}{GM}$$

Remember that this only works for a *fluid* body!

- Does this make sense?
- Why is this expression useful?
- Is it reasonable to assume a fluid body?

Pause & Summary

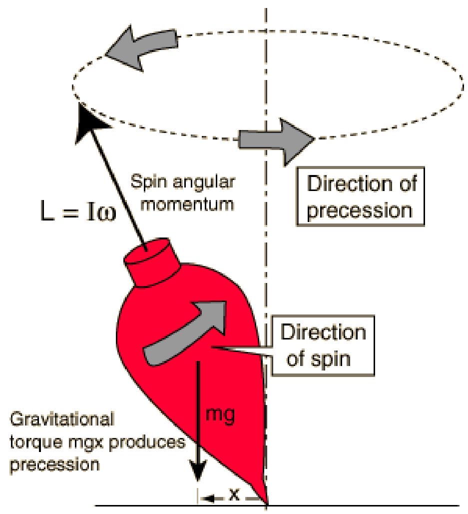
- Moment of inertia depends on distribution of mass
- For planets, $C>A$ because mass is concentrated at the equator as a result of the rotational bulge
- The gravity field is affected by the rotational bulge, and thus depends on C-A (or, equivalently, J_2)
- So we can measure C-A remotely (e.g. by observing a satellite's orbit)
- If the body has no elastic strength, we can also predict the *shape* of the body given C-A (or we can infer C-A by measuring the shape)

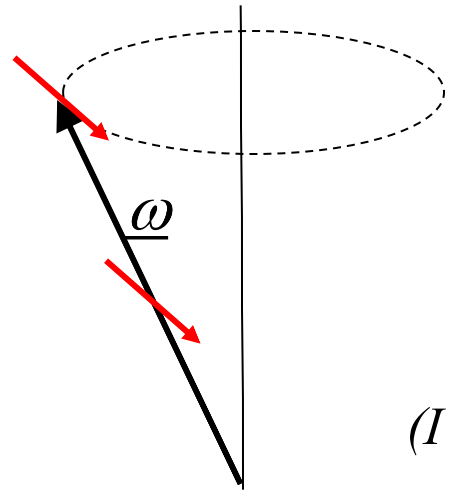
How do we get *C* from *C-A*?

- Recall that we can use observations of the gravity field to obtain a body's MoI difference *C-A*
- But what we would really like to know is the actual moment of inertia, *C* (why?)
- Two possible approaches:
  - Observations of precession of the body's axis of rotation
  - *Assume* the body is fluid (hydrostatic) and use theory

Precession (1)

- Application of a *torque* to a rotating object causes the rotation axis to move in a circle - *precession*



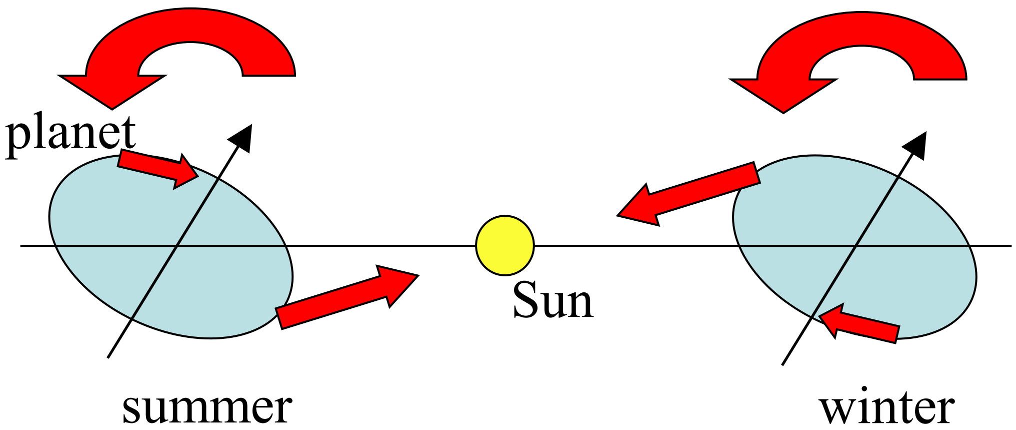


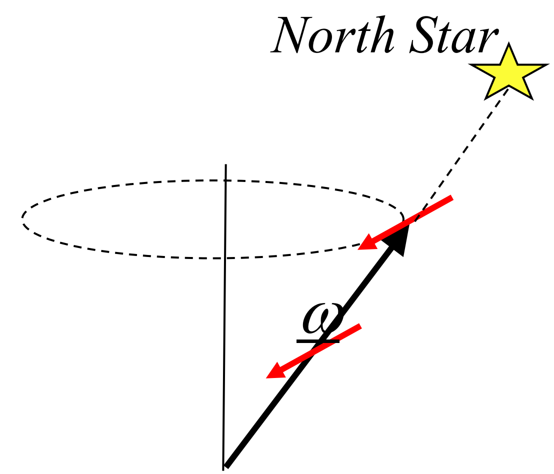
$$\underline{T}=I\, d\underline{\omega}/dt$$

(*I* is moment of inertia)

- The circular motion occurs because the instantaneous torque is perpendicular to the rotation axis
- The *rate of precession* increases with the torque $\underline{T}$, and decreases with increasing moment of inertia (*I*)
- An identical situation exists for rotating planets . . .

Precession (2)





- So the Earth's axis of rotation also precesses
- In a few thousand years, it will no longer be pointing at the North Star
- The rate of precession depends on torque and MoI (C)
- The torque depends on C-A (why?)
- So the rate of precession gives us $(C\text{-}A)/C$

Putting it together

- If we can measure the rate of precession of the rotation axis, we get $(C\text{-}A)/C$
- For which bodies do we know the precession rate?
- Given the planet's gravitational field, or its flattening, we can deduce J_2 (or equivalently $C\text{-}A$)
- Given $(C\text{-}A)/C$ and $(C\text{-}A)$, we can deduce C
- Why is this useful?
- What do we do if we can't measure the precession rate?

Hydrostatic assumption

- In most cases, the precession rate of the planet is not available. How do we then derive *C*?
- If we assume that the planet is *hydrostatic* (i.e. it has no elastic strength), then we can derive *C* directly from *C*-*A* using the Darwin-Radau approximation*:

$$f_{hyd} = \frac{\frac{5}{2}\frac{\omega^2 a^3}{GM}}{1+(\frac{25}{4})(1-\frac{3}{2}\frac{C}{Ma^2})^2}$$

Here the flattening depends on *C*. We also have an equation giving *f* in terms of *C*-*A* (see before)

- This tells us the flattening expected for a fluid rotating body with a non-uniform density distribution
- Does this equation make sense?

** I'm not going to derive this; see C.R. Acad. Sci. Paris 100, 972-974, 1885 and Mon. Not. R. Astron. Soc. 60 82-124 1899*

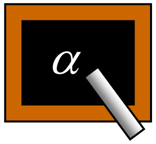
Earth as an example

- We can measure J_2 and thus calculate f (assuming a fluid Earth): J_2=1.08x10^{-3} , $\omega^2 a^3/GM$=3.47x10^{-3}

$$f=(3/2)J_2 + \omega^2 a^3/2GM$$

$$\therefore \quad f=3.36\text{x}10^{-3}$$

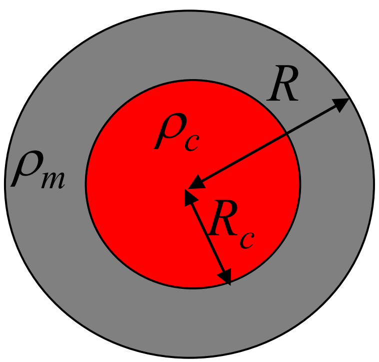
- The observed flattening f=3.35x10^{-3}. Comments?
- A fluid Earth is a good assumption, so we can use f and the Darwin-Radau relation to obtain C/Ma^2
- We get an answer of C/Ma^2=0.331. The real value is 0.3308.

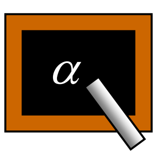


- What do we conclude?
- Next: what use is knowing C/Ma^2, anyway?

What use is C/MR^2?

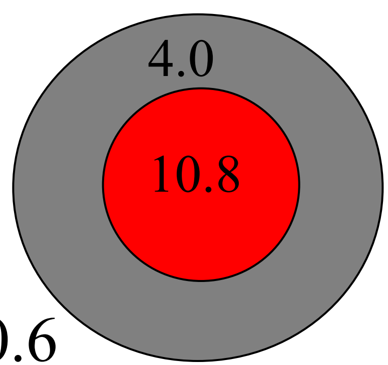
- We have two observations: M (or ρ_{bulk}) and C/MR^2
- For a simple two-layer body, there are three unknowns: mantle and core densities, and core radius
- If we specify one unknown, the other two are determined
- E.g. if we pick ρ_m, r' $(=R_c/R)$ and ρ_c can be calculated





For instance, Earth ρ_{bulk}=5.55 g/cc, C/MR^2=0.33

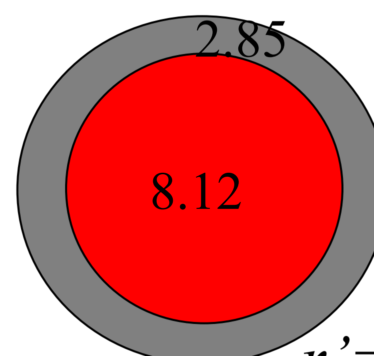
For ρ_m=4 g/cc we get:



r'=0.6

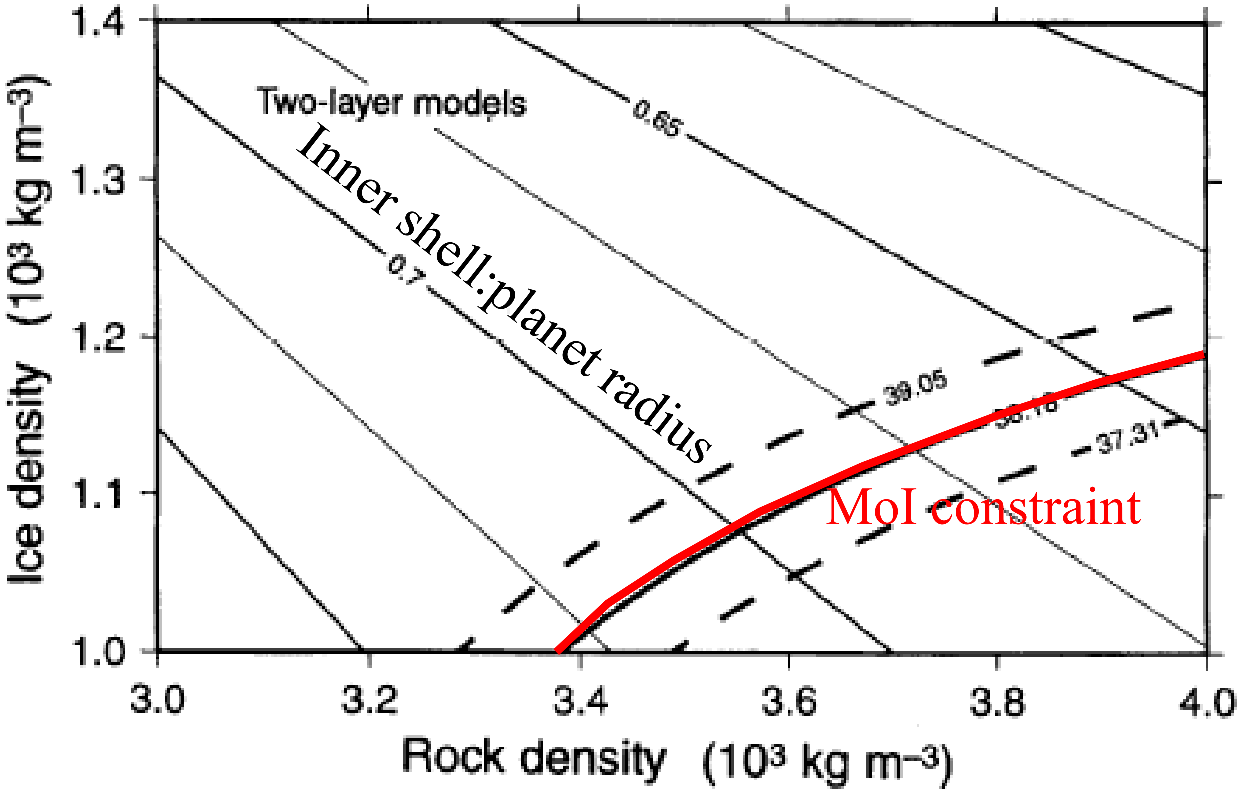
Which model is more likely?

For ρ_m=2.85 g/cc we get:

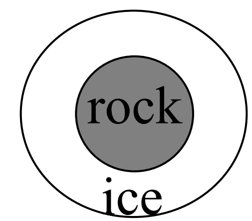


$r'=0.8$

Example - Ganymede



Anderson et al., *Nature* 1996



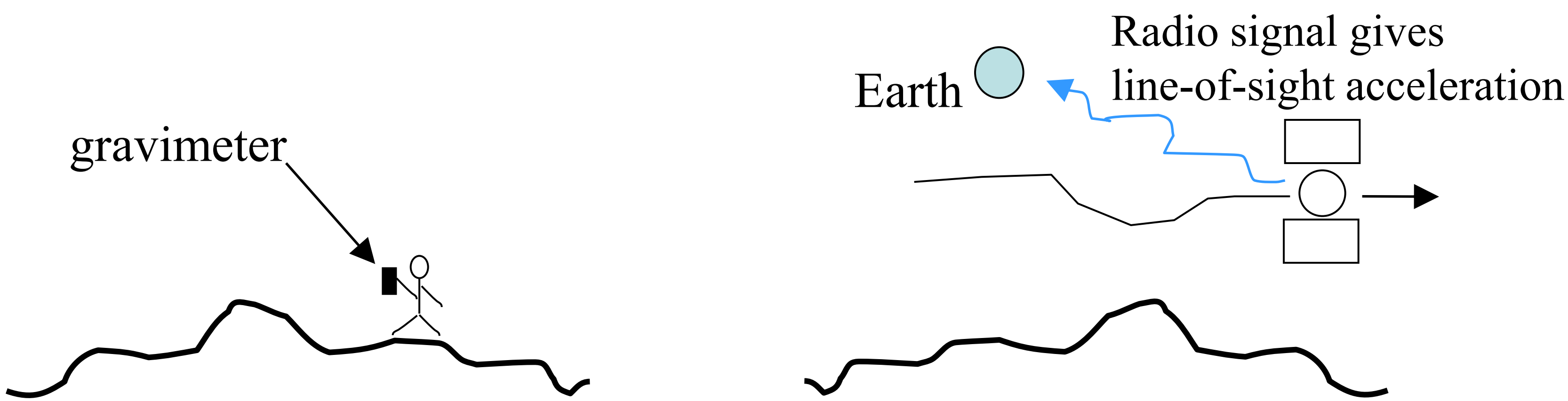
- Two-layer models satisfying mass and MoI constraints
- Again, if we specify one unknown (e.g. rock density), then the other two are determined
- Here C/MR^2=0.31 – mass v. concentrated towards the centre

Pause & Summary

- Measuring the gravity field gives us J_2 (or C-A)
- To get the internal structure, we need C
- Two options:
  - 1) Measure the *precession rate of the rotation axis* of the body (requires a lander). The precession rate depends on $(C-A)/C$, so we can deduce C
  - 2) Assume that the body is hydrostatic. This allows us to deduce C directly from C-A
- We normally express C as C/MR^2, for comparison with a uniform sphere (C/MR^2=0.4)
- Most bodies have $C/MR^2 < 0.4$, indicating a concentration of mass towards their centres (differentiation)

Local gravity variations (1)

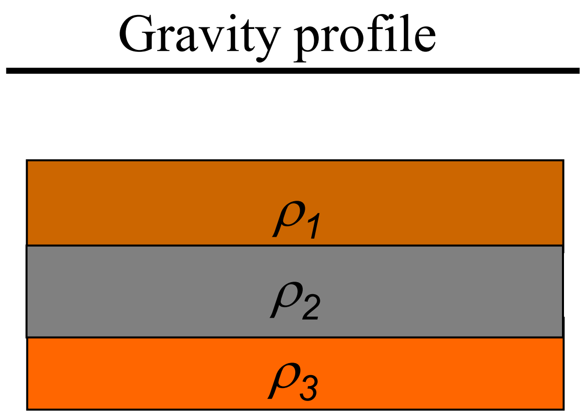
- So far we have talked about using planet-scale variations in gravity to infer bulk structure
- We can also use more local observations of gravity to make inferences about local subsurface structure
- How do we make such local observations?

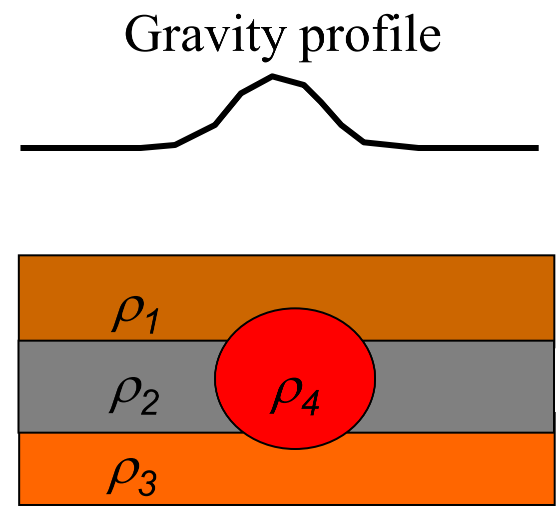


- Local gravity anomalies are typically small, so we use units of milliGals (mGal). 1 mGal=10^{-5} ms^{-2}~10^{-6} g_{Earth}

Local gravity variations (2)

- Local variations in the gravity field arise from *lateral variations* in the density structure of the subsurface:





- The magnitude of the gravity anomaly depends on the size of the body and the density contrast (see later)
- The magnitude of the anomaly also depends on how the observer is above the anomaly source (gravity falls off with distance)

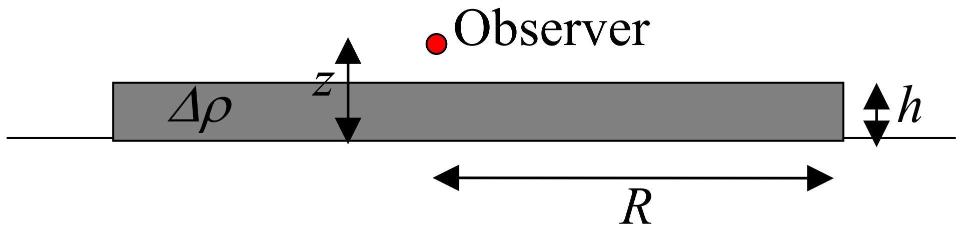
Free Air Gravity

- When making ground-based observations, we have to correct for our latitude (MacCullagh's formula)
- We also have to correct for the fact that the local gravity we measure depends on our *elevation* (as well as any local anomalies)
- This correction is known as the *free-air correction* and gives us the gravity as if measured at a constant elevation
- The free air correction Δg for an elevation h is given by

$$\Delta g = \frac{2hg_0}{R}$$

Here g_0 is the reference accleration due to gravity and R is the planetary radius
This correction is only correct for $h \ll R$

Gravity due to a plate



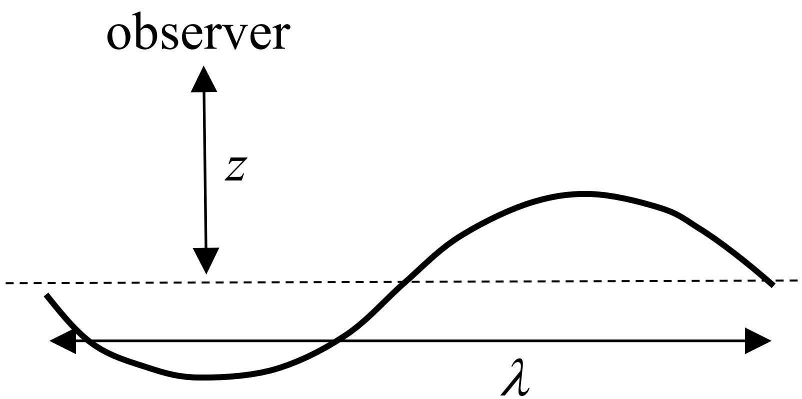
- For an observer close to the centre ($z<<R$) of a flat plate of thickness h and lateral density contrast $\Delta\rho$, the gravity anomaly Δg is simply:

$$\Delta g = 2\pi \Delta\rho h G$$

- A useful number to remember is that this equation gives 42 mGals per km per 1000 kg m^{-3} density contrast
- This allows us to do things like e.g. calculate the gravitational anomaly caused by the Himalayas (see later)

Attenuation

- The gravity that you measure depends on your distance to the source of the anomaly
- The gravity is *attenuated* at greater distances



- The attenuation factor is given by **exp(-*kz*)**, where $k=2\pi\backslash\lambda$ is the *wavenumber* (see T&S eq. 5-123)

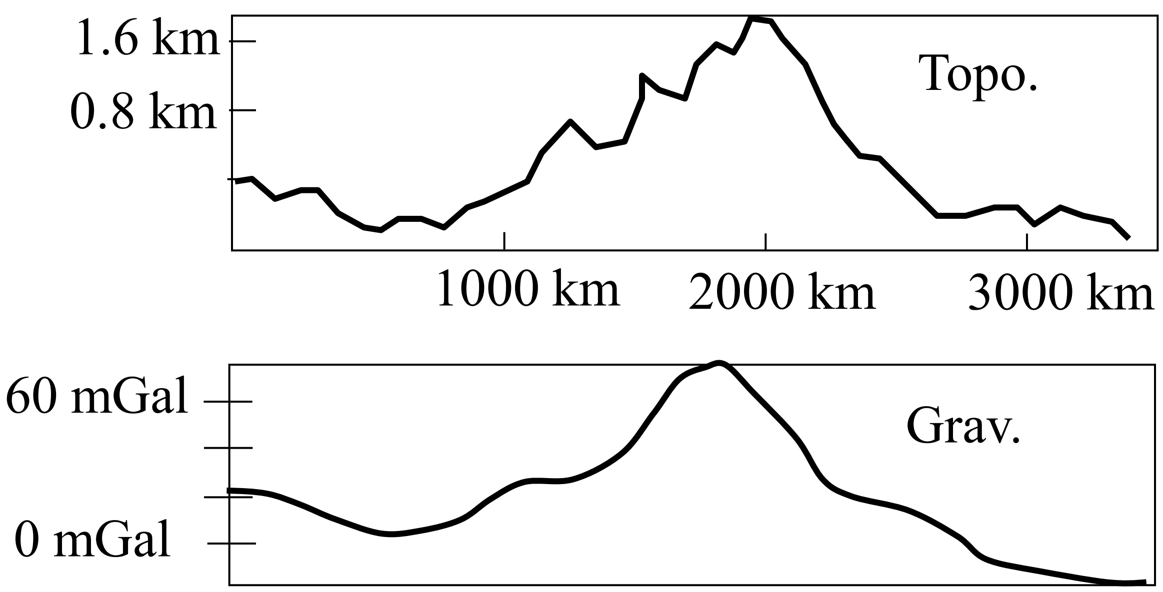
- What does this mean? Short wavelength signals are attenuated at lower altitudes than longer-wavelength ones

Most gravity calculations can be done using just attenuation and the plate formula!

Example - Venus

- What acceleration would we see at spacecraft altitude?
- How does this compare with what we actually see?
- What is the explanation for the discrepancy?

Spacecraft altitude 200 km, topo wavelength ~2000 km



Note that the gravity signal is much smoother than the topo – why?

Nimmo & McKenzie, *EPSL* 1996

Summary

- *Global* gravity variations arise due to MoI difference (J_2)
- So we can measure $J_2=C-A$ remotely
- We can also determine C, either by observation or by making the hydrostatic assumption
- Knowing C places an additional constraint on the internal structure of a planet (along with density)
- *Local* gravity variations arise because of lateral differences in density structure
- We can measure these variations by careful observation of a spacecraft's orbit
- The variations are *attenuated* upwards, depending on the observation altitude and wavelength

Homework #2

- Is posted, due next Monday