Warmup (NPC)

1 a) Find the general solution for the following differential equation: [3]

\[
\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 4y = 0
\]

b) Find the general solution for the following differential equation [3]:

\[
\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 13y = 0
\]

c) Find the particular solution for the following differential equation, subject to the boundary conditions that \( y = 0 \) at \( x = 0 \) and \( \frac{dy}{dx} = -3 \) at \( x = 0 \): [4]

\[
\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0
\]

d) Find the general solution for the following differential equation: [4]

\[
\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = \sin x
\]

2 This is designed to give you some practice with the techniques we’ve developed for solving differential equations. Let’s say we have a second-order, homogeneous differential equation

\[
\frac{d^2y}{dx^2} + f \frac{dy}{dx} = 0 \tag{1}
\]

where \( f \) is a constant.

a) Write down the characteristic polynomial and find the two roots of \( m \). [2]

b) Hence write down the general solution of this equation [1]

For this particular equation (but not in general!) we can use a different solution technique by setting \( z = \frac{dy}{dx} \).

c) Rewrite equation (1) as a first-order differential equation in \( z \) [1]

d) The resulting equation is separable. Write down the general solution for \( z \). [2]

e) Now that you have \( z \), find \( y \). Is the answer that you get the same as in b)? [2]
Now we’ll add a forcing term, so that our equation is non-homogeneous:

\[
\frac{d^2y}{dx^2} + f \frac{dy}{dx} = a \sin kx
\]  

where \(a\) and \(k\) are constants.

f) Find the extra solution arising from the forcing term. You should assume an equation of the form \(y = A \sin kx + B \cos kx\) and solve for \(A\) and \(B\) in terms of \(a, k\) and \(f\). [5]

g) Hence find the general solution of equation (2) [1] [14 total]