Warmup (NPC)

1) Find the general solutions to the following second-order constant coefficient homogeneous differential equations:
   a) \( y'' + 4y' + 4y = 0 \)
   b) \( y'' + 6y' + 13y = 0 \)
   [4 total]

2) Find the full solution to the following HLDE:
   \( y'' - 3y' - 4y = 0 \)
   subject to the boundary conditions that \( y' = -3/2 \) and \( y = 1 \) at \( x = 0 \).
   [3 total]

3) Find the general solutions to the following second-order constant coefficient nonhomogeneous differential equations:
   a) \( y'' - 3y' - 4y = 6x^2 \)
   b) \( y'' + y' - 2y = e^x \)
   [5 total]

4) A seismometer is an example of a damped, forced simple harmonic oscillator. The general equation for the displacement \( x \) recorded by the seismometer can be written
   \[
   \frac{d^2x}{dt^2} + \frac{2c}{m} \frac{dx}{dt} + \frac{k}{m}x = F_0\omega_0^2 \cos \omega_0 t
   \]
   where \( m \) is the mass, \( c \) a damping coefficient, \( k \) the spring constant, \( t \) is time and the forcing (e.g. an earthquake) has a constant amplitude \( F_0 \) and a constant frequency \( \omega_0 \).

   We’re going to assume that the seismometer is critically damped, which means that \( c^2 = mk \).

   a) Solve the complementary equation for this seismometer, assuming that it’s critically damped. Your solution should have two undetermined constants, \( c_1 \) and \( c_2 \). [3]

   b) Now we’re going to think about the particular solution. Assume it’s of the form \( x = A \cos \omega_0 t + B \sin \omega_0 t \) and hence find \( A \) and \( B \) in terms of \( F_0 \) and the other parameters.
**Hints:** Your algebra will be simpler if you replace $c/m$ with $\omega$ and $k/m$ with $\omega^2$, where $\omega$ is the natural (undamped) frequency of the seismometer. And both $A$ and $B$ should have denominators of $(\omega_0^2 + \omega^2)^2$. [8]

c) The amplitude of the seismometer response $R$ is given by $R = \sqrt{A^2 + B^2}$. Using your answer to b), show that the response $R$ is given by

$$R = \frac{F_0}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

[4]

d) Sketch how the response $R$ varies with the forcing frequency $\omega_0$ for a given value of $F_0$ and $\omega$. What is the value of the response at $\omega = \omega_0$? [3]

e) Is a seismometer of this kind better at recording high-frequency or low-frequency signals? Why? [2] [20 total]