Warmup (NPC)
1a) Solve the following differential equation:
\[
\frac{dN}{dt} = -ktN
\]
subject to the boundary condition that \( N = N_0 \) at \( t = 0 \). Here \( k \) is a constant. [3]
b) The outwards orbital evolution of a satellite is given by
\[
\frac{da}{dt} = \beta a^{-1/2}
\]
where \( a \) is the distance of the satellite from the planet, \( \beta \) is a constant for a particular planet-satellite pair, and \( t \) is time.

This is a separable differential equation. Write down the general solution [3]
c) Assuming that \( a = a_0 \) at \( t = 0 \), write down the particular solution [2]. [8 total]

2 The InSight lander on Mars has detected interfaces at mid-crustal depths. Here we’re going to explore whether this could be the result of pores being squeezed out.

We can write an expression for the closure of porosity \( \phi \) as follows:
\[
\frac{d\phi}{dt} = -\frac{P}{\eta} \phi
\]
Here \( P \) is the overburden pressure (Pa) and \( \eta \) is the viscosity of the rock (Pa s).

a) If the viscosity of the rock is lower, do you expect the pores close faster or slower? [1]
b) Taking both \( P \) and \( \eta \) to be constant, find the general solution to equation (1) (i.e. find an expression for \( \phi(t) \)). [2]
c) We’ll apply the boundary condition that \( \phi = \phi_0 \) (a constant) at \( t = 0 \). Use this boundary condition to find the particular solution for \( \phi(t) \). [2]
d) What units does the quantity \( \eta/P \) have? What does this represent physically? [2]

A possible complication is that porosity might be being created (e.g. by impacts) as well as destroyed. So we’ll modify equation (1) to:
\[
\frac{d\phi}{dt} = -\frac{P}{\eta} \phi + \beta e^{-kt}
\] 

(2)

where \( \beta \) describes the rate of porosity production and the \( e^{-kt} \) term reflects the fact that impact bombardment declines with time.

e) After a very, very long time, how much porosity do you expect to be left, and why? [1]

f) Using an integrating factor, find the general solution to equation (2) [3]

g) Using the boundary condition \( \phi = \phi_0 \), find the particular solution [3].

h) If \( \beta = 0 \), what does your solution look like, and does this make sense? [1]

i) If \( t \to \infty \), what value of \( \phi \) do you get? [1]

j) If the viscosity is too high, the pores won’t close at all. In this case, what do you expect to happen to the porosity over time? [1]

k) For a high enough viscosity, \( P/\eta \to 0 \). What does your answer to g) simplify to? Sketch how the porosity changes with time in this case [3] [20 total]