Warmup (NPC)

1 a) Solve the following differential equation:

\[ \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 6y = 0 \]

subject to the boundary conditions that \( y = 0 \) at \( x = 0 \) and \( dy/dx = 5 \) at \( x = 0 \).

b) Solve the following differential equation:

\[ \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 5y = 0 \]

subject to the boundary conditions that \( y = 0 \) at \( x = 0 \) and \( y = 1 \) at \( x = \pi/2 \).

2 One of the classic applications of differential equations is elastic flexure.

The governing equation in the case of an elastic lithosphere is

\[ D \frac{d^4 w}{dx^4} + \rho gw = 0 \]  

where \( x \) is the horizontal position, \( w(x) \) is the deflection of the lithosphere, \( D \) is its flexural rigidity (in Nm), \( \rho \) is the density and \( g \) is the gravity. This is a fourth-order, homogeneous DE with constant coefficients.

a) One possible solution to this DE is

\[ w = ae^{-mx} \cos mx \]  

where \( a \) is an undetermined constant and \( m \) is determined by the lithospheric characteristics. By substituting this solution into the governing DE, find an expression for \( m \) in terms of \( \rho, g \) and \( D \). [Note that I am not asking you to solve equation (1) directly; I’m just asking you to show that equation (2) is a solution of equation (1)].

b) The quantity \( mx \) must be dimensionless and \( x \) has the units of length. Does your answer to a) make sense in terms of units?

c) Sketch what equation (2) looks like for \( x > 0 \). What controls how far away from the origin there is a noticeable deflection? What would happen if the rigidity \( D \) were increased?
So far we have dealt with a homogeneous equation (the right-hand side of equation 1 is zero). But we can also analyze what happens if there is a driving term e.g. a load applied to the lithosphere. Let’s say there is a periodic load applied given by

\[ l = \rho h_0 \sin kx \]

where \( h_0 \) and \( k \) are constants with units of m and m\(^{-1}\), respectively. This is now the right-hand side of the governing DE (equation 1).

d) Now assume that a solution to the DE is given by

\[ w = w_0 \sin kx \]

where the deflection amplitude \( w_0 \) is a constant controlled by the lithospheric and load characteristics. By substituting into the governing DE, find a relationship between \( w_0 \) and \( h_0 \) in terms of \( k, g, \rho \) and \( D \). [4]

e) What happens to the amplitude of the deflection \( w_0 \) compared to the amplitude of the load \( h_0 \) if the rigidity \( D \) is increased? Does this make sense? [2]

f) What happens to the amplitude of the deflection \( w_0 \) compared to the amplitude of the load \( h_0 \) if \( k \) is decreased? Does this make sense? What does \( k \) represent? [3] [19 total]

3 The outwards orbital evolution of a satellite is given by

\[ \frac{da}{dt} = \beta a^{-11/2} \]

where \( a \) is the distance of the satellite from the planet, \( \beta \) is a constant for a particular planet-satellite pair, and \( t \) is time.

a) This is a separable differential equation. Write down the general solution [3]

b) Assuming that \( a = a_0 \) at \( t = 0 \), write down the particular solution [2].

c) Using your answer to b), sketch roughly how \( a \) varies with \( t \). [2]

d) If \( a \) has not changed much from the initial value of \( a_0 \), show that \( a \) depends in a linear fashion on \( t \). Hint: it will help to write \( a \) as \( a_0 + \delta a \), where \( \delta a \) is small. [3] [11 total]