Warmup (NPC)

1 a) Solve the following differential equation:

\[ \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 6y = 0 \]

subject to the boundary conditions that \( y = 0 \) at \( x = 0 \) and \( \frac{dy}{dx} = 5 \) at \( x = 0 \) [4].

b) Solve the following differential equation:

\[ \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 5y = 0 \]

subject to the boundary conditions that \( y = 0 \) at \( x = 0 \) and \( y = 1 \) at \( x = \pi/2 \) [4] [8 total].

2 Cosmic rays bombarding silica occasionally knock off some protons and neutrons from the oxygen or silicon nuclei, producing radiogenic isotopes of aluminium and beryllium. Because these nuclides decay with time, geologists can use them to estimate erosion rates. Here’s the theory. The production of the nuclides is a function of the burial depth \( z \) (how far below the surface the silica is), the penetration depth \( l \) (how deep the cosmic rays can reach), and a constant \( P_0 \) specific to the isotope in question. Mathematically, the production rate is given by

\[ P_0 e^{-z/l} \]

Since the cosmogenic nuclides are unstable, they decay over time at a rate given by

\[ kC(t) \]

where \( k \) is the decay constant and \( C(t) \) is the concentration of nuclides in the sample at that time.

Combining the above two effects, we get

\[ \frac{dC}{dt} = P_0 e^{-z/l} - kC \]  \hspace{1cm} (1)

Note that here \( z \) is a constant.

a) Obtain the general solution to this differential equation. [3]

b) Obtain the particular solution given that \( C = 0 \) at \( t=0 \). [2]
c) Sketch this solution (i.e. show $C$ as a function of $t$). What is the steady state value of concentration $C(t)$? (that is, what does $C(t)$ tend to for large values of $t$?). Explain the physical reason for the answer that you get. [4]

Now let’s add in erosion by making $z$ a function of time $t$:

$$z = E(\tau - t) \quad (2)$$

where $E$ is the erosion rate (units of velocity), $E\tau$ is the initial depth of burial (at time $t=0$) and both $E$ and $\tau$ are constants.

d) Substitute (2) for $z$ in equation (1) and obtain the general solution of the resulting differential equation. [4]

e) Obtain the particular solution given that the initial condition $C(0) = 0$ at $t = 0$. [1]

f) Write down an expression for the concentration of radionuclides in the sample when it first reaches the surface (i.e. when $t = \tau$). You may assume that the initial depth of burial is sufficiently large that $e^{-E\tau/l} \to 0$. [2]

g) Using f), write down an expression for the erosion rate $E$ in terms of $P_0$, $k$, $l$ and $C$, the measured concentration. If you obtain two samples (A and B) which have similar ages ($\tau$), and A has a higher nuclide concentration than B, then which sample was exhumed more rapidly, and why? Does your answer make physical sense? [3] [19 total]

3 The outwards orbital evolution of a satellite is given by

$$\frac{da}{dt} = \beta a^{-11/2}$$

where $a$ is the distance of the satellite from the planet, $\beta$ is a constant for a particular planet-satellite pair, and $t$ is time.

a) This is a separable differential equation. Write down the general solution [3]

b) Assuming that $a = a_0$ at $t = 0$, write down the particular solution [2].

c) Using your answer to b), sketch roughly how $a$ varies with $t$. [2]

d) If $a$ has not changed much from the initial value of $a_0$, show that $a$ depends in a linear fashion on $t$. Hint: it will help to write $a$ as $a_0 + \delta a$, where $\delta a$ is small. [3] [11 total]