Warmup (NPC)

1 a) Find the eigenvectors and eigenvalues of the following matrix, and hence sketch the resulting strain ellipse [5]:

\[
\begin{pmatrix}
3 & 1 \\
1 & 3 \\
\end{pmatrix}
\]

b) Solve the following differential equation:

\[
\frac{dN}{dt} = -ktN
\]

subject to the boundary condition that \( N = N_0 \) at \( t = 0 \). Here \( k \) is a constant. [3] [8 total]

2a) Write down a matrix representing a rotation by \( \theta \) in the clockwise direction [1]

b) Write down a matrix representing simple shear by a factor \( \gamma \) in a direction parallel to the \( y \)-axis [1]

c) Write down the matrix representing the combined effects of the rotation followed by the simple shear [2]

d) Find an expression for the eigenvalue(s) \( \lambda \) of this matrix. You can leave it as a quadratic in \( \lambda \) [3]

e) If this quadratic has repeated roots (i.e. \( \lambda_1 = \lambda_2 = \lambda \)), then there are only two possible values of \( \lambda \). What are they? [2]

f) Use your answer to e) to show that if the roots are repeated, then the following is true [3]:

\[
\gamma = \frac{2(\pm1 - \cos \theta)}{\sin \theta}
\]

(1)

g) If there are repeated roots, that means there is only a single eigenvector. Explain physically how the case of a single eigenvector arises. [2] [14 total]

3 Here we are going to consider the pressure in a gas envelope surrounding a massive planet.

The pressure gradient is given by

\[
\frac{dP}{dr} = -\frac{P \mu GM}{r^2 R_g T}
\]

(2)
where $P$ is pressure, $r$ is radial distance from the centre of the planet, and $\mu, G, M, R_g$ and $T$ are constants representing the gas molar mass, gravitational constant, planet mass, gas constant and gas temperature, respectively.

a) Find the general solution to equation (2). [3]

b) The boundary condition is that at the surface of the planet ($r = R_s$) the pressure is $P_s$. Find the particular solution to equation (2). (Write it as an exponential). [3]

c) At small radial distances we can replace $r$ with $R_s + z$, where $z$ is height above the surface. If $z$ is sufficiently small, show that we can approximate $P$ as

$$P(z) \approx P_s \exp \left( -\frac{\mu g_s}{R_g T} z \right)$$

where $g_s = GM/R_s^2$ is the surface gravity. [3]

d) At very large radial distances, the pressure approaches an asymptotic value. What is this value? Why is that a peculiar result? [2]

e) Part of the reason that we obtain this peculiar result is that one of our “constants” is probably not really constant. Which is the most likely candidate? [1] [12 total]