Warmup (NPC)

1a) If we have a matrix \( A \), where

\[
A = \begin{pmatrix}
1 & 2 \\
-1 & 3
\end{pmatrix}
\]

then write down \( A^T \), \( AA \), \( |A| \) and \( A^{-1} \). [4]

b) Using Gauss-Jordan elimination, find \( B^{-1} \) where:

\[
B = \begin{pmatrix}
4 & 1 & 1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}
\]

[4] [8 total]

2) a) Given two matrices \( A \) and \( D \), what does \((AD)^T\) equal in terms of \( A^T \) and \( D^T \)? [1]

b) Using your answer to a), what does \((ABCD)^T\) equal in terms of \( A^T \), \( B^T \) and \( C^T \)? [2]

c) Here we are going to use two special kinds of matrices: a symmetrical matrix \( A \), where \( A = A^T \); and an orthogonal matrix \( B \), where \( B^T = B^{-1} \).

We’ll define a matrix \( D = B A B^{-1} \).

By taking the transpose of \( D \) and using your answer to b), show that \( D \) is also a symmetrical matrix. [4]

d) Also prove that the product of two orthogonal matrices is also orthogonal. [3]

[10 total]

3) Here we’re going to use linear algebra to fit a line through a set of points.

Let’s say we have a set of three points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\).

If all these points lie on a single line then \( y_i = mx_i + c \) with \( m \) and \( c \) constants.

In this case we can write down the relationship between the \( y_i \) and \( x_i \) as a matrix equation:

\[
A x = b
\]
Here
\[ x = \begin{pmatrix} c \\ m \end{pmatrix} \]
and
\[ b = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \]

a) Write down the matrix \( A \). [3]

In reality, the points will not all lie on the line. But it turns out that we can still find the best-fit line by setting
\[ A^T (Ax - b) = 0 \]

b) This expression can be rewritten \( Cx = d \). Write down \( C \) and \( d \) in terms of \( A \), \( A^T \) and \( b \). [3]

c) For the three points (1, 0.8), (2, 2.1) and (3, 2.9), write down the matrices \( C \) and \( d \). [5]

We can solve the equation \( Cx = d \) using Gaussian elimination or simultaneous equations.

d) Use your answer to c) to find the best-fit values of \( m \) and \( c \) for the three points. [4]

e) Reality check: does your answer make sense? (Hint: look at the \( x, y \) values again). [1] [16 total]