Warmup (NPC)

1a) If we have a matrix $A$, where

$$A = \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}$$

then write down $A^T$, $AA$, $|A|$ and $A^{-1}$. [4]

b) Using Gauss-Jordan elimination, find $B^{-1}$ where:

$$B = \begin{pmatrix} 4 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

[4] [8 total]

2) a) Given two matrices $A$ and $D$, what does $(A \ D)^T$ equal in terms of $A^T$ and $D^T$? [1]

b) Using your answer to a), what does $(A \ B \ C)^T$ equal in terms of $A^T$, $B^T$ and $C^T$? [2]

c) Here we are going to use two special kinds of matrices: a symmetric matrix $A$, where $A = A^T$; and an orthogonal matrix $B$, where $B^T = B^{-1}$.

We’ll define a matrix $D = B \ A \ B^{-1}$.

By taking the transpose of $D$ and using your answer to b), show that $D$ is also a symmetric matrix. [4]

d) Also prove that the product of two orthogonal matrices is also orthogonal. [3]

[10 total]

3) One of the uses of matrices is in seeing how stresses in a rock vary with orientation. Here we’ll use two matrices. The first one describes the stresses acting on a rock

$$S = \begin{pmatrix} x & b \\ b & y \end{pmatrix}$$

Here $x$ and $y$ are the (normal) stresses acting in the X and Y directions, and $b$ is the shear stress.
The second matrix has the form

\[
R = \begin{pmatrix}
c & s \\
-s & c
\end{pmatrix}
\]

where you can just treat \(c\) and \(s\) as constants for now.

a) We want to calculate a new matrix \(S'\) which is given by

\[
S' = R \cdot S \cdot R^T
\]

where \(^T\) denotes transpose. Find the components of the matrix \(S'\) in terms of \(c, s, x, y\) and \(b\). [5]

b) We want the off-diagonal components of the matrix \(S'\) to be zero. Write down an expression in terms of \(c, s, x, y\) and \(b\) that describes this condition. [1]

If we now set \(c = \cos \theta\) and \(s = \sin \theta\), then equation (??) calculates the stresses resolved onto a plane which is at an angle \(\theta\) to the X-axis.

c) Setting \(c = \cos \theta\) and \(s = \sin \theta\), use your answer to b) to show that the angle at which the off-diagonal components are zero is given by

\[
\tan 2\theta = \frac{2b}{x - y}
\]

Note that you will need to use double angle formulae here. [4]

d) Using your answers to a) and c), show that the top left component of \(S'\) is given by [4]

\[
\frac{(x + y)}{2} + \frac{(x - y)}{2} \cos 2\theta + b \sin 2\theta
\]

e) Your answer to d) is the component of (normal) stress in the \(\theta\)-direction. What does this answer reduce to if \(\theta = 0\) or \(\theta = \pi/2\)? Does this behaviour make sense? [3] [17 total]