Warmup (NPC)

1a) If \( f(x, y) = 2x^2 + xy^2 + y \) then write down \( \nabla f \) [2]

b) For the same function, find the maximum slope at \((0,1)\) [1]

c) For the same function, what is the slope at \((0,1)\) in the \([0,1]\) direction? [1]

d) For the same function, find the location of the critical point [2]

e) Find the location of the four critical points of the function given below. (\textbf{Hint:} think about how to decompose \( \ln(xy) \)) [2]

\[
f(x, y) = 4x^2 + y^2 - \ln(xy)
\]

[8 total]

2) Given the equation

\[
z = f(x, y) = \frac{x^4}{4} - \frac{2x^3}{3} - \frac{3x^2}{2} + 2y^2
\]

a) Sketch this curve in the \( x - z \) plane (i.e. set \( y=0 \)). This includes identifying the \( x \) and \( z \) values at which the curve crosses the \( x \) and \( z \) axes, identifying the \( x \) and \( z \) values at which turning points occur, and showing what happens as \( x \to \pm\infty \). [5]

b) Sketch this curve in the \( y - z \) plane [1]

c) Identify where the critical point(s) of the surface are. [3]

d) For the three critical points on the \( x \)-axis, find their nature (e.g. max, min etc.) [3]

e) If we set \( y = x \), we’ll get a profile of the surface along the line \( y = x \). This profile has two turning points. At what values of \( x \) do these turning points occur? [2] [14 total]

3) Let’s say that we want to fit a set of data \((x_i, y_i)\) to a power-law curve passing through the origin:

\[
y_i' = (x_i)^a
\]

where \( y_i' \) is our model prediction and \( a \) is the parameter that we need to determine.

One way of doing this is to use the least squares technique after taking logs of both sides.

a) We’re going to define some new variables: \( q_i = \ln x_i \), \( p_i = \ln y_i \) and our model prediction \( p_i' = \ln y_i' \). Rewrite equation (1) in terms of \( p_i' \), \( q_i \) and \( a \). [1]
b) Write down an expression for the mismatch \( E_i = (p_i - p'_i) \) between an observed and predicted data point, in terms of \( p_i, a \) and \( q_i \). [1]

c) The mismatch for an individual point is \( E_i \), but we want to minimize the total squared mismatch \( E_1^2 + E_2^2 + E_3^2 + \cdots = \sum E_i^2 \).

Write down an expression for \( \sum E_i^2 \) in terms of \( p_i, a \), and \( q_i \). [2]

d) To minimize the mismatch, we take the derivative with respect to \( a \) (normally it would be a partial derivative, but here we only have one variable). Write down an expression for \( \frac{d \sum E_i^2}{da} \). [2]

e) Now set the derivative to zero, and solve for \( a \). Your answer will make use of \( \sum q_i^2 \) and \( \sum p_i q_i \). [3]

f) If all the data points lie exactly on the curve, show that your answer to e) gives the correct answer. [2]

g) Below is a set of data. Using the results you developed above, find the best-fit value of \( a \) [hint: don’t forget to take the logs!]. Suggest one way in which you might improve the fit. [3] [14 total]

Table 1:

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( y_i )</th>
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