Answer **Question 1** and **three (3)** of Questions 2-6.  
You have 1 hour and 35 mins. to complete the test.  
No calculators, phones or cheat sheets to be used (except the Formula Sheet I’ve provided you with).  
Make sure you show *all* your working!

**Question 1** [Everyone must answer this question]

a) Find the following integral [2]:

\[ \int x^2 e^x \, dx \]

b) Use the double angle formulae to write down an expression for \( \tan(2\theta) \) in terms of \( \tan \theta \). [2]

c) Find the first three non-zero terms in the Maclaurin series expansion of \( (1 - x)^n \). [2]

d) If \( \hat{a} \) and \( \hat{b} \) are unit vectors with an angle \( \gamma \) between them, write down a simple expression for \((\hat{a} \otimes \hat{b}) \cdot (\hat{b} \otimes \hat{a})\). [2]

e) Write down the equation of the plane passing through the points \((2,1,0), (1,1,1)\) and \((3,2,1)\). [2]

f) Given the following function

\[ f(x, y) = 2x^2 y + xy \]

write down \( \nabla f \) and \( \nabla^2 f \). [3]

\[ \begin{align*}
g) & \text{ Write down one thing you like about the course, and one thing that could be improved [1]. [14 total]}
\end{align*} \]

**Question 2**

You are given the function

\[ f(x, y) = 2y^3 - xy^2 + x \]

a) Find the location of the critical points, and identify whether they are max, min or saddle [4]

b) Sketch the function along the line \( y = -x \). This includes finding where the horizontal axis is crossed and identifying where the turning points are, and whether they are max or min. [4]

**Question 3**

a) The wave equation may be written

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

where \( u \) is displacement and \( c \) is speed (a constant).

By assuming that equation (1) can be satisfied by a solution of the form

\[ u(x, t) = u_0 \sin(kx + \omega t) \]

determine the relationship between \( c, \omega \) and \( k \). Here \( u_0, k \) and \( \omega \) are constants. [5]
b) The physical meanings of \( k \) and \( \omega \) are \((2\pi/\text{wavelength})\) and \((2\pi/\text{period})\), respectively. Hence write down an expression for the speed \( c \) in terms of wavelength and period. Does this make physical sense? [3] [8 total]

**Question 4**

Here we want to fit a circle passing through the origin to a set of \( N \) points \((x_i, y_i)\) (see figure).

![Figure 1](image)

a) Write down the distance \( d_i \) of the point \((x_i, y_i)\) from the origin. [1]

b) We’ll define the mismatch between the observed and predicted data point as \( E_i = d_i - r \), where \( r \) is the radius of the circle.

We want to vary the circle radius \( r \) so the total squared mismatch \( E_1^2 + E_2^2 + \cdots + E_N^2 = \Sigma E_i^2 \) is minimized.

Write down an expression for \( \Sigma E_i^2 \) in terms of \( x_i, y_i \) and \( r \) [2]

c) To minimize the mismatch, we take the derivative with respect to \( r \). Write down an expression for \( \frac{\Sigma E_i^2}{dr} \). [2]

d) Now set the derivative to zero, and solve for \( r \). Your answer will depend on \( \Sigma(x_i^2 + y_i^2)^{1/2} \). [2]

e) Hence write down in words how you calculate the best-fit radius. [1]

**Question 5**

Consider a vector field \( \mathbf{v} = [yz, -xz, xy] \).

a) Write down \( \nabla \otimes \mathbf{v} \) [3]

b) Also write down \( \nabla \cdot \mathbf{v} \) [1]

c) What does \( \nabla \cdot (\nabla \otimes \mathbf{v}) \) equal? [2]

d) Also write down a scalar field \( f \) which has the property that \( \nabla f = \nabla \otimes \mathbf{v} \) [2]

**Question 6**

a) Write down the first two non-zero terms in the Maclaurin series expansion for \( \cos(x) \). [2]

b) Do the same thing for \( \sin(x) \). [2]

c) Now write down the first four terms in the Maclaurin series expansion for \( e^{ix} \). Here \( i \) is just a constant with the special property that \( i^2 = -1 \). [3]

d) Hence show that \( e^{ix} = \cos(x) + i \sin(x) \). [1] [8 total]