Answer Question 1 and three (3) of Questions 2-6.

You have 1 hour and 35 mins. to complete the test.

No calculators, phones or cheat sheets to be used (except the Formula Sheet I’ve provided you with).

Make sure you show all your working!

**Question 1 [Everyone must answer this question]**

a) Write down the first three terms in the Maclaurin series expansion of \((1-x)^n\). [2]

b) The angle between two vectors \(a\) and \(b\) is \(\gamma\). Find an expression for \((a \otimes b) \cdot (b \otimes a)\) in terms of \(|a|, |b|\) and \(\gamma\). [2]

c) Find the equation for the plane that passes through the points (3,2,1), (0,3,3) and (1,1,1). [2]

d) The function \(f(x,y)\) is given by \(f(x,y) = x^2 + 2xy + 2y\). Find the location of the critical point and determine what kind of point it is. [3]

e) Using the definition of differentiation, prove that if \(f(x) = x^2\), then \(f'(x) = 2x\). [2]

f) The function \(T(x,y)\) is given by \(T(x,y) = \cos(ax) \sin(by)\). If it is also true that
\[
\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = 0
\]

then find the relationship between \(a\) and \(b\). [2]

g) Write down one thing you like about the course, and one thing that could be improved. [1]

[14 total]

**Question 2**

a) Write down the first two terms in the Maclaurin series expansion of \(\frac{1}{1+x}\). [2]

b) Use your answer to a) to show that \(e^{\frac{1}{1+x}}\) can be approximated by \((1 - ax)e^a\) when \(x\) is small. [3]

c) The viscosity \(v_0\) of a fluid is given by \(v_0 = Ae^{Q/T_0}\) where \(A\) and \(Q\) are constants and \(T_0\) is a particular temperature.

The temperature now changes from \(T_0\) by a small amount \(\delta\). Using the results above, show that the viscosity at this new temperature is approximately given by \(v_0 \left(1 - \frac{Q\delta}{T_0^2}\right)\). [4] [9 total]

**Question 3**

a) The vector field \(\vec{F}\) is given by \(\vec{F} = [cy, -cx, 0]\) where \(c\) is a constant. Find the value of \(c\) that makes the following relationship true: \(\nabla \otimes \vec{F} = [0, 0, 1]\). [3]

b) For any position vector \(\vec{v} = (x, y, 0)\), what is the dot product \(\vec{v} \cdot \vec{F}\), and what does this imply about the two vectors? [2]

c) How does the magnitude of the field \(|\vec{F}|\) vary with \(r\), the distance from the origin? [2]

d) Sketch a couple of contours of the field in the \(x, y\) plane. [2] [9 total]
**Question 4**

The function \( T(x, y) \) is given by \( T(x, y) = e^{ax} \sin by \).

a) If \( \nabla^2 T = 0 \), find the relationship between \( a \) and \( b \). [2]

b) Find the location of the critical points of this function, if any. [1]

c) At the origin, what is the magnitude and direction of the steepest slope for this function? [3]

d) Find an approximate expression including both \( x \) and \( y \) for \( T(x, y) \) at a point close to the origin (i.e. when \( x \) and \( y \) are both small). [3]

**Question 5**

You are given the function \( f(x, y) = x^3 - \frac{x}{2} + xy + y^2 \).

a) Find the two critical points of this function and determine what kind of points they are. [5]

b) Sketch the function along the line \( x = -2y \). This includes finding at least one point where the horizontal axis is crossed, and identifying where the turning points are. [4] [9 total]

**Question 6**

a) For a point at latitude \( \theta_1 \), longitude \( \phi_1 \) on a sphere, what are the \( x \), \( y \) and \( z \) components of the unit position vector describing this point? [3]

b) If there is a second point at latitude \( \theta_2 \), longitude \( \phi_2 \), show that the angle between these two position vectors \( \gamma \) is given by [3]

\[
\cos \gamma = \cos \theta_1 \cos \theta_2 \cos(\phi_1 - \phi_2) + \sin \theta_1 \sin \theta_2
\]

c) We will define a third position vector which is perpendicular to the two vectors above. Show that the \( z \)-component of this vector is given by \( \cos \theta_1 \cos \theta_2 \sin(\phi_2 - \phi_1) \). [2]

d) Explain geometrically why the \( z \)-component is zero if \( \phi_1 = \phi_2 \). [It may help to draw a sketch] [1] [9 total].