Fields

We have spent a lot of time thinking about functions of the form \( z = f(x, y) \). For instance, a topographic surface is a set of points with elevations \( z \) determined by their \( x \) and \( y \) positions. This surface is an example of a **scalar field**, where a scalar quantity (in this case, elevation) varies as a function of position. Other common examples of scalar fields include pressure, temperature, and gravitational potential (potential is a scalar quantity).

However, there are also situations in which the quantity we are mapping is a **vector**, e.g. a map of wind velocities. If a vector quantity varies as a function of position, then it is called a **vector field**. Typical examples include gravitational acceleration and fluid velocities. You can think of a vector field as a collection of arrows distributed across a map.

A general expression describing a vector field is \( \mathbf{v} = \left[ f_1(x, y), f_2(x, y) \right] \), where \( \mathbf{v} \) is a vector which has \( x \) and \( y \) components \( f_1 \) and \( f_2 \), both of which may be functions of position. Vector fields can also be 3D, but these are harder to draw.

**Example:** \( \mathbf{v} = [x^2 + y^2, x^2 - y^2] \) is an example of a vector field. **What does it look like?**

Operators

An **operator** is something that acts on a function. We have already met operators before. For instance

\[
\frac{d}{dx}
\]

is an operator that acts on a function \( f(x) \) to produce the first derivative:

\[
\frac{d}{dx} f(x) = \frac{df}{dx}
\]

Operators can be scalars or vectors. In the example above, the operator \( \frac{d}{dx} \) is a scalar. Operators aren’t very meaningful on their own; they are meaningful when they are applied to some function.

It turns out that the gradient of a function, \( \nabla f \), also involves an operator.
Recall that

\[ \nabla f = \text{grad } f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] \]  

(1)

Here \( f(x, y) \) is a scalar field.

Also recall that for any vector \( \mathbf{v} = [x, y] \) then \( h\mathbf{v} = [hx, hy] \) where \( h \) is some constant.

So we can rewrite equation (1) as

\[ \nabla f = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] f \]  

(2)

Written this way, we can see that \( \nabla \) is a vector operator, where \( \nabla = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] \).

In three dimensions, we write

\[ \nabla = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \]

You can think of \( \nabla \) as behaving a bit like a vector, although as with \( \frac{d}{dx} \) it only has real meaning when applied to some function. Equation (2) shows that multiplying a vector (\( \nabla \)) with a scalar (\( f \)) yields a vector, as required.

**Divergence**

The gradient of \( f \), \( \nabla f \), is the result of applying a vector operator \( \nabla \) to a scalar field \( f \), and gives us a vector.

If we think of \( \nabla \) as behaving like a vector, then we can apply it to a vector field in two ways. **What are they?**

One way is to take the dot product. If we have a vector field \( \mathbf{v} = [f_1, f_2, f_3] \) then we write the dot product of this field with the gradient operator as

\[ \nabla \cdot \mathbf{v} = \text{div } \mathbf{v} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \]

Notice that the dot product of two vectors has yielded a scalar, as required. This scalar quantity is called the *divergence* of the function.

**Example** What is \( \nabla \cdot \mathbf{v} \) when \( \mathbf{v} = [x^2 + y^2, x^2 - y^2, xy] \)?

[Answer: \( 2x - 2y \)]

**What use is the divergence?** It tells you about the total flow of some quantity (heat, material etc.) into or out of a particular region.
For instance, the total amount of material flowing into or out of a box is given by \( \nabla \cdot \mathbf{v} \), where \( \mathbf{v} \) is a velocity field and we’ve assumed that density is constant. Situations in which \( \nabla \cdot \mathbf{v} = 0 \) everywhere are called **incompressible** and are generally easier to deal with than compressible cases.

Similarly, if we consider a small box, the total amount of heat flowing into or out of the box depends on \( \nabla \cdot \mathbf{F} \), where \( \mathbf{F} \) is a vector field describing the heat flux. Why? What does this imply for the temperature field?

**Answer:** We learnt before that \( \mathbf{F} = -k \nabla T \) where \( T \) is the temperature field and \( k \) the thermal conductivity. So the total amount of heat flowing in or out depends on \( \nabla \cdot \nabla T \).

**Curl**

The other way of combining two vectors is to use the cross product.

So we can cross the gradient operator \[ \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \] with some vector function \( \mathbf{v} = [f_1, f_2, f_3] \) to get the **curl** of \( \mathbf{v} \):

\[
\nabla \times \mathbf{v} = \text{curl } \mathbf{v} = \left[ \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right]
\]

**Example** What is \( \nabla \times \mathbf{v} \) when \( \mathbf{v} = [x^2 + y^2, x^2 - y^2, xy] \)?

**Answer:** \([x, -y, 2x - 2y]\)

**What use is curl?** It crops up a lot in electromagnetics (Maxwell’s equations). Physically, you can think of the curl of a vector field as telling us the **axis of rotation** of the field - think of a little paddle wheel sitting in the field. (How do you show this?) A field for which \( \nabla \times \mathbf{v} = 0 \) everywhere is called **irrotational**. What is an example of irrotational flow?

**Note that** while \( \nabla \cdot \mathbf{v} \) is defined in 2D or 3D, \( \nabla \times \mathbf{v} \) is only meaningful in 3D.

**Combining operators**

If you think about operators as behaving like vectors, you can see that they can be combined in various ways.

For instance, you can show (how?) that

\[
\nabla \times \nabla f = 0
\]

where \( f \) is a scalar field.

What about

\[
\nabla \times (\nabla \cdot \mathbf{v})
\]
where $\mathbf{v}$ is a vector field?

Finally, what do you get if you write out

$$\text{div grad } f = \nabla \cdot \nabla f$$

The quantity div grad $f$ is called the Laplacian and is often written $\nabla^2 f$. Where have you seen this before?

[Answer: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$. This shows up in diffusion problems.]

Maxwell’s equations are good examples of where vector operators are used. Two of the four equations are as follows:

$$\nabla \otimes \mathbf{B} = \mu_0 \mathbf{J}$$

(assuming a steady current) and

$$\nabla \cdot \mathbf{B} = 0$$

where $\mathbf{B}$ and $\mathbf{J}$ are the magnetic field and current density, respectively, and $\mu_0$ is a constant.

**Coordinate Systems**

All the results in this lecture are given for Cartesian geometries. Div, grad and curl are always physically meaningful, but the expressions for calculating them are different in different geometries. Any textbook will give the expressions for div, grad and curl in spherical and cylindrical geometries, as well as Cartesian.

**Reminders**

1. PS5 is due this evening (6pm).