Directional Derivatives and the Gradient

You can think of the function $z = f(x, y)$ as a contour plot e.g. of temperature around an intrusion. The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ can be interpreted as the rate of change (or slope) of $f(x, y)$ in the $x$ and $y$ directions, respectively. For our intrusion example, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ would be the temperature gradients in the $x$ and $y$ directions.

We can also evaluate derivatives along directions intermediate between $x$ and $y$ (or $\hat{i}$ and $\hat{j}$ in vector notation). The result is a directional derivative, which simply tells us the value of the derivative in a particular direction.

All directional derivatives are weighted sums of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. The weights depend on the direction.

Let $\hat{u}$ be a unit vector in two dimensions, giving us the direction in which we wish to take the derivative:

$$\hat{u} = [a, b], \quad |\hat{u}| = 1$$

The directional derivative of $f$ in the direction specified by $\hat{u}$ is denoted $D_u f$ and is given by

$$D_u f = a \left( \frac{\partial f}{\partial x} \right) + b \left( \frac{\partial f}{\partial y} \right) = af_x + bf_y$$

This is a scalar quantity (it’s just a slope). You can see by inspection that this gives the correct answer if $\hat{u}$ is parallel to either the $x$ or $y$ axis.

Example What is $D_u f$ when $f(x, y) = x^2 + y^2$ and $\hat{u} = [1,0],[0,1]$ and $\frac{1}{\sqrt{2}}[1, 1]$?

For most functions $f(x, y)$, at each point $(x, y)$ there is usually one direction for which the magnitude of the directional derivative is greatest i.e. the function is changing most rapidly. (Think about being on a hill-side - there is usually one direction of greatest slope).

Example In what direction does the greatest rate of change happen at the point $(1, 1)$ when $f(x, y) = x^2 + y^2$?

The direction of greatest rate of change and its magnitude are obtained from the gradient of $f$:

$$\nabla f = \hat{i} \left( \frac{\partial f}{\partial x} \right) + \hat{j} \left( \frac{\partial f}{\partial y} \right) = \hat{f}_x + \hat{f}_y = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$
The gradient of $f$ is denoted $\nabla f$ and often called “del $f$”. Although $f$ is a scalar (it has magnitude, but no direction), the gradient of $f$ is a vector. Note that this vector lies in the $x-y$ plane.

The gradient has special properties:

1. $\nabla f$ points in the direction of the greatest rate of change of $f$. For instance, if $f$ describes a temperature field, then the gradient of $f$ gives the $[x, y]$ vector that points in the direction of greatest rate of temperature change.

2. $\nabla f$ points in a direction perpendicular to contours of $f(x, y)$.

3. The magnitude of the greatest rate of change (i.e. the maximum slope) is just $|\nabla f|$.

4. $D_u f = \hat{u} \cdot \nabla f$. In other words, we can calculate the directional derivative in any direction $\hat{u}$ by taking the dot product of $\hat{u}$ with the gradient of $f$ (why does this work?).

The gradient is very important. Many physical systems involve diffusion down a gradient - heat, charge, composition etc. Water always tries to flow down the steepest slope available. Gradients are also important in optimization problems.

Note that the analysis given here is for functions of two variables $f(x, y)$. You can derive similar approaches for functions of three or more variables, but it becomes harder to visualize what’s happening.

Also note that the analysis given here is for Cartesian co-ordinates. The gradient also exists for other co-ordinate systems (e.g. polar - $r, \theta$) but has a slightly different form.

Example Say we have a function defined by

$$f(x, y) = e^{-(x^2+y^2)}$$

What does this function look like? What is the vector $\nabla f$? What direction is the steepest slope pointing in at $(1,0)$, $(0,1)$ and $(1,1)$?

Example Let’s now look at our bowl-shaped function from last lecture:

$$f(x, y) = 2x^2 + y^2$$

What direction is the steepest slope along the $x = y$ line? Does this make sense?

Example The flow of fluid through a porous medium is given by D’Arcy’s law:

$$u = -\frac{k}{\eta} \nabla P$$
where \( \mathbf{u} \) is the fluid velocity, \( k \) is the permeability, \( \eta \) is the viscosity and \( P(x, y) \) is the pressure. Is this physically reasonable? Is it dimensionally correct?

If the pressure field is given by
\[
P = \frac{1}{x^2 + y^2}
\]
then what does the fluid flow vector look like? Does this make sense?

The heat flux \( \mathbf{E} \) through a conductive material has the same form:
\[
\mathbf{E} = -k \nabla T
\]
where now \( k \) is the thermal conductivity and \( T \) is temperature. Note that here heat flux is being treated as a vector quantity, which is correct: the heat is flowing in a particular direction (down the temperature gradient).

**Example - gravity**

In 2D, the gravitational potential \( U \) (a scalar quantity) at a position \((x, y)\) of a point mass \( M \) located at the origin is given by
\[
U = -\frac{GM}{(x^2 + y^2)^{1/2}}
\]
where \( G \) is the gravitational constant. Find a vector expression for the resulting gravitational acceleration \( \mathbf{g} \), where \( \mathbf{g} = -\nabla U \). (Note that \( \nabla \) converts a scalar into a vector).

\[
\mathbf{g} = -\nabla U = -\left[ \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} \right]
\]

Writing \( r^2 = x^2 + y^2 \) we have
\[
\frac{\partial U}{\partial x} = GM \frac{x}{r^3}, \quad \frac{\partial U}{\partial y} = GM \frac{y}{r^3}
\]
so that
\[
\mathbf{g} = -\frac{GM}{r^3} [x, y]
\]

Is this correct? First, what direction does \( \mathbf{g} \) point in? Second, what is the magnitude of \( \mathbf{g} \), that is \( |\mathbf{g}| \)?

Problem Set 3 is due this evening