So far we have dealt with functions of the form $y = f(x)$, where $x$ is the independent variable and $y$ is the dependent variable whose value is determined by $x$. More generally, we would expect $y$ to depend on several independent variables $y = f(x, y, z, \ldots)$. We would like to extend calculus to functions of several variables, which involves the use of partial derivatives.

**Where are partial derivatives useful?** Everywhere. Partial derivatives are completely inescapable in any real physical system, since there is always more than one variable which affects the outcome. A typical example is cooling problems (e.g. a magmatic intrusion) where the temperature $T$ depends on position $T = f(x, y)$ (and also depends on time).

We will start with two independent variables

$$z = f(x, y)$$

Here $x$ and $y$ are independent variables (we can assign them any values we choose) and $z$ is the dependent variable. A good example of a function of two variables is a plane (see last lecture):

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

which we can rewrite

$$z = \frac{1}{c}[-a(x - x_0) - b(y - y_0)] + z_0 = f(x, y)$$

The vertical coordinate ($z$) is a function of the $x$ and $y$ positions. In our dike example above, $z$ would represent the temperature.

Functions of one variable are easy to graph - they make a curve in 2D (plot $y$ against $x$). Functions of two variables are *surfaces* in 3D - a plane is just a particularly uninteresting surface. One way of visualizing the surface is to calculate the $x, y$ positions for which $z$ stays the same - this is a contour line.

Graphing functions of 2 variables is difficult, because we have to represent 3 dimensions on a 2D sheet of paper, and we have to let both independent variables vary. A good way to start is to determine the curves in the $x = 0$ and $y = 0$ planes.
**Example**

What does the surface

\[ z = 2x^2 + y^2 \]

look like?

Start by setting \( y = 0 \) and drawing the curve in the \( x - z \) plane, given by

\[ z = 2x^2 \]

This is a parabola.

![Figure 1](image)

Next we set \( x = 0 \) and draw the curve in the \( y - z \) plane, given by

\[ z = y^2 \]

This is also a parabola, but with a slightly different shape.

![Figure 2](image)

Finally, we can think about what values of \( x \) and \( y \) produce the same value of \( z \) - this is equivalent to finding *contour lines* for the function. To do this, set \( z = k \), where \( k \) is the value of the contour line we are interested in. In this case, all the contour lines are ellipses

\[ 2x^2 + y^2 = k \]

with intercept values of \( x = \pm \sqrt{k/2} \) and \( y = \pm \sqrt{k} \), respectively.
So the overall surface has the shape of a bowl, elongated in the $y$ direction.

We are not limited to functions of two variables. In fact, calculus can handle functions of arbitrary numbers of variables. To make the notation simple, we often use subscripts, such that

$$y = f(x_1, x_2, x_3, \ldots, x_n)$$

represents a function of $n$ variables. We can simplify this further using vector notation. Letting

$$\mathbf{x} = [x_1, x_2, x_3, \ldots, x_n]$$

Then we can write the function as

$$y = f(\mathbf{x})$$

**Partial Derivatives**

Let $z = f(x, y)$. How does $z$ change when we vary $x$? How does $z$ change when we vary $y$? We answer these questions by extending the concept of a derivative to functions of several variables. By analogy with the derivative of $y = f(x)$:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

we look at the limit

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

which we refer to as the *partial derivative* of $f$ with respect to $x$. What we are doing here is **holding $y$ fixed** and finding the slope of the curve $f(x, y)$ in the $x$-direction. Similarly,

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

is the partial derivative of $f$ with respect to $y$, that is, the slope of the curve in the $y$-direction with $x$ held fixed. Notice the different symbols used for partial derivatives.
Why is it called a partial derivative? Because it describes variation in only one degree of freedom. Functions of two or more variables have two or more degrees of freedom; the partial derivative is only examining the sensitivity of the function to one particular degree of freedom. You can think of \( \frac{\partial f}{\partial x} \) as giving the slope of the function in the \( x \)-direction and \( \frac{\partial f}{\partial y} \) as the slope of the function in the \( y \)-direction.

We obtain partial derivatives just as we do regular derivatives, treating the ignored independent variable(s) as constants. For example

\[
f(x, y) = x^2 + 2y^2
\]

has partial derivatives

\[
\frac{\partial f}{\partial x} = 2x \\
\frac{\partial f}{\partial y} = 4y
\]

The product rule works the same way as for regular derivatives.

**Example:** \( f(x, y) = e^{xy} \sin(xy) \)

The chain rule can be applied to partial derivatives in many ways. Some important examples include the following, for a function \( z = f(g(x, y)) \):

\[
\frac{\partial z}{\partial x} = \left( \frac{\partial}{\partial g} z \right) \left( \frac{\partial g}{\partial x} \right)
\]

and

\[
\frac{\partial z}{\partial y} = \left( \frac{\partial}{\partial g} z \right) \left( \frac{\partial g}{\partial y} \right)
\]

**Example:** \( z = e^{\sin(xy)} \)

Let \( g = \sin(xy) \) so that \( z = e^g \). Then \( \frac{\partial z}{\partial x} = y \cos(xy) \) and \( \frac{\partial z}{\partial y} = e^g \). So

\[
\frac{\partial z}{\partial x} = y \cos(xy) e^{\sin(xy)}
\]

**Example:** \( z = e^{ax^2+by^2} \)

Higher derivatives follow the same rules

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} \\
\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \frac{\partial f}{\partial y}
\]
\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} \]

This last expression is due to Clairaut and says that the order of partial differentiation is not important. This also extends to higher derivatives, e.g.:

\[ \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2} \]

**Examples**

Find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) for the following:

1. \( f(x, y) = xy \)
2. \( f(x, y) = x \cos y \)
3. \( f(x, y) = (x^2 + y^2)^{1/2} \)
4. \( f(x, y) = \cos(xy) \)
5. \( f(x, y) = \cos((x^2 + y^2)^{1/2}) \)

In steady-state with no heat sources, the 2D thermal diffusion equation may be written (where does this come from?)

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = 0 \]

Show that an equation of the form

\[ T = T_0 \exp \left( \frac{-2\pi z}{\lambda} \right) \sin \left( \frac{2\pi x}{\lambda} \right) \]

satisfies this relationship.

I don’t have time to discuss it here, but this is an important physical result: variations in the \( x \)-direction with short wavelength (small \( \lambda \)) only propagate a small distance in the \( z \)-direction (and vice versa).

**Total Differential**

**Where is this useful?** If forms the basis of error analysis.

For functions of 1 variable, the regular derivative is also the total differential. That is, for \( y = f(x) \)

\[ dy = \frac{df}{dx} \, dx \]
is a total differential. Here \( dy \) describes the variation of the function about a point. Another way of saying it is that \( dy \) tells you how much \( y \) varies as you make small changes \( dx \) in \( x \) - a concept closely related to the uncertainty or error in \( y \) (of which more later).

Functions of several variables also have total differentials. For \( z = f(x, y) \) the total differential \( dz \) is given by

\[
dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy
\]

As before, \( dz \) approximates the change in \( z \) for small increments \((dx, dy)\) in \( x \) and \( y \).

**Error Analysis**

Let’s say that \( z = f(x, y) \) and the uncertainties (errors) in \( x \) and \( y \) are \( dx \) and \( dy \), respectively. Then \( dz \), the expected error in \( z \), is given by

\[
|dz| = \left| \frac{\partial f}{\partial x} dx \right| + \left| \frac{\partial f}{\partial y} dy \right|
\]

where here we are employing absolute values (||) to avoid opposite signs cancelling the error.

**Example** Say we want to estimate the shear modulus of a material from its density and shear velocity. If the values obtained for the density and shear velocity are 2700 ± 40 kg m\(^{-3}\) and 3500 ± 100 m s\(^{-1}\), respectively, what is the error in the resulting shear modulus? (Here we are using \( v_s^2 = \mu/\rho \)).

**Problem Set 3** is due this Friday (14th)