So far we have dealt with functions of the form \( y = f(x) \), where \( x \) is the independent variable and \( y \) is the dependent variable, whose value is determined by \( x \). More generally, we would expect \( y \) to depend on several independent variables \( y = f(x, y, z, \cdots) \). We would like to extend calculus to functions of several variables, which involves the use of partial derivatives.

**Where are partial derivatives useful?** Everywhere. Partial derivatives are completely inescapable in any real physical system, since there is always more than one variable which affects the outcome. A typical example is the gravity anomaly of a buried object, where the measured gravity anomaly \( g \) will depend on both the height and horizontal distance of the observer, that is \( g = f(x, y) \).

We will start with two independent variables

\[
z = f(x, y)
\]

Here \( x \) and \( y \) are independent variables (we can assign them any values we choose) and \( z \) is the dependent variable. A good example of a function of two variables is a plane (see last lecture):

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0
\]

which we can rewrite

\[
z = \frac{1}{c}[-a(x - x_0) - b(y - y_0)] + z_0 = f(x, y)
\]

Once we have specified \( x \) and \( y \), the quantity \( z \) is then determined by the equation. In our example above \( z \) would represent the gravity anomaly, which depends on \( x \) and \( y \).

Functions of one variable are easy to graph - they make a curve in 2D (plot \( y \) against \( x \)). Functions of two variables are surfaces in 3D - a plane is just a particularly simple surface. One way of visualizing the surface is to calculate the \( x, y \) positions for which \( z \) stays the same - this is a contour line.

Graphing functions of 2 variables is difficult, because we have to represent 3 dimensions on a 2D sheet of paper, and we have to let both independent variables vary. A good way to start is to determine the curves in the \( x = 0 \) and \( y = 0 \) planes, that is to graph \( y \) against \( z \), and \( x \) against \( z \).
Example

What does the surface

\[ z = 2x^2 + y^2 \]

look like?

Start by setting \( y = 0 \) and drawing the curve in the \( x - z \) plane, given by

\[ z = 2x^2 \]

This is a parabola.

![Figure 1](image)

Next we set \( x = 0 \) and draw the curve in the \( y - z \) plane, given by

\[ z = y^2 \]

This is also a parabola, but with a slightly different shape.

![Figure 2](image)

Finally, we can think about what values of \( x \) and \( y \) produce the same value of \( z \) - this is equivalent to finding contour lines for the function. To do this, set \( z = k \), where \( k \) is the value of the contour line we are interested in. In this case, all the contour lines are ellipses

\[ 2x^2 + y^2 = k \]

with intercept values of \( x = \pm \sqrt{k/2} \) and \( y = \pm \sqrt{k} \), respectively.
So the overall surface has the shape of a bowl, elongated in the $y$ direction.

We are not limited to functions of two variables. In fact, calculus can handle functions of arbitrary numbers of variables. To make the notation simple, we often use subscripts, such that

$$y = f(x_1, x_2, x_3, \ldots, x_n)$$

represents a function of $n$ variables. We can simplify this further using vector notation. Letting

$$x = [x_1, x_2, x_3, \ldots, x_n]$$

Then we can write the function as

$$y = f(x)$$

**Partial Derivatives**

Let $z = f(x, y)$. How does $z$ change when we vary $x$? How does $z$ change when we vary $y$? We answer these questions by extending the concept of a derivative to functions of several variables. By analogy with the derivative of $y = f(x)$:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

we look at the limit

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

which we refer to as the partial derivative of $f$ with respect to $x$. What we are doing here is holding $y$ fixed and finding the slope of the curve $f(x, y)$ in the $x$-direction. Similarly,

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$
is the partial derivative of \( f \) with respect to \( y \), that is, the slope of the curve in the \( y \)-direction with \( x \) held fixed. Notice the different symbols used for partial derivatives.

Why is it called a partial derivative? Because it describes variation in only one degree of freedom. Functions of two or more variables have two or more degrees of freedom; the partial derivative is only examining the sensitivity of the function to one particular degree of freedom. You can think of \( \frac{\partial f}{\partial x} \) as giving the slope of the function in the \( x \)-direction and \( \frac{\partial f}{\partial y} \) as the slope of the function in the \( y \)-direction.

We obtain partial derivatives just as we do regular derivatives, treating the ignored independent variable(s) as constants. For example

\[
f(x, y) = x^2 + 2xy^2
\]

has partial derivatives

\[
\frac{\partial f}{\partial x} = 2x + 2y^2
\]

\[
\frac{\partial f}{\partial y} = 4xy
\]

The product rule works the same way as for regular derivatives.

**Example:** \( f(x, y) = e^{xy} \sin(xy) \)

[Answer: \( \frac{\partial f}{\partial x} = ye^{xy} \sin(xy) + e^{xy}y \cos(xy) \), \( \frac{\partial f}{\partial y} = xe^{xy} \sin(xy) + e^{xy}x \cos(xy) \)].

The chain rule can be applied to partial derivatives in many ways. Some important examples include the following, for a function \( z = f(g(x, y)) \):

\[
\frac{\partial z}{\partial x} = \left( \frac{\partial}{\partial g} z \right) \left( \frac{\partial g}{\partial x} \right)
\]

and

\[
\frac{\partial z}{\partial y} = \left( \frac{\partial}{\partial g} z \right) \left( \frac{\partial g}{\partial y} \right)
\]

**Example:** \( z = e^{\sin(xy)} \)

Let \( g = \sin(xy) \) so that \( z = e^g \). Then \( \frac{\partial g}{\partial x} = y \cos(xy) \) and \( \frac{\partial g}{\partial y} = x \sin(xy) \). So

\[
\frac{\partial z}{\partial x} = y \cos(xy)e^{\sin(xy)}
\]

**Example:** \( z = e^{ax^2+by^3} \)

[Answer: \( \frac{\partial z}{\partial x} = 2axe^{ax^2+by^3} \), \( \frac{\partial z}{\partial y} = 3by^2e^{ax^2+by^3} \).]
Higher derivatives follow the same rules

\[ \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \]

\[ \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) \]

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \]

This last expression says that if you differentiate a function with respect to \( x \), and then differentiate the result with respect to \( y \), you will get the same answer as if you reversed the order of operations.

This same rule also extends to higher derivatives, e.g.:

\[ \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2} \]

**Example** If the function is \( f(x, y) = x^2 + 2y + xy \) then \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1 \).

**Examples**

Find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) for the following:

1. \( f(x, y) = xy \)
2. \( f(x, y) = x \cos y \)
3. \( f(x, y) = (x^2 - y^2)^{1/2} \)
4. \( f(x, y) = \cos(xy) \)
5. \( f(x, y) = \cos((x^2 - y^2)^{1/2}) \)

[Answers:

1. \( y, x \).
2. \( \cos y, -x \sin y \).
3. \( x(x^2 - y^2)^{-1/2}, -y(x^2 - y^2)^{-1/2} \).
4. \( -y \sin(xy), -x \sin(xy) \).
5. \( -x(x^2 - y^2)^{-1/2} \sin(x^2 - y^2)^{1/2}, y(x^2 - y^2)^{-1/2} \sin(x^2 - y^2)^{1/2} \).]

In steady-state with no heat sources, the 2D thermal diffusion equation may be written (**where does this come from?**)

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = 0 \]

Show that an equation of the form

\[ T = T_0 \exp \left( -\frac{2\pi z}{\lambda} \right) \sin \left( \frac{2\pi x}{\lambda} \right) \]
satisfies this relationship.

I don’t have time to discuss it here, but this is an important physical result: variations in the $x$-direction with short wavelength (small $\lambda$) only propagate a small distance in the $z$-direction (and vice versa).

**Total Differential**

**Where is this useful?** If forms the basis of error analysis.

For functions of 1 variable, the regular derivative is also the total differential. You can think of the total differential as telling you how much $y$ changes for some change in $x$. That is, for $y = f(x)$

$$dy = \frac{df}{dx} \, dx$$

is a total differential. The quantity $dy$ tells you how much $y$ varies as you make small changes $dx$ to $x$ - a concept closely related to the uncertainty or error in $y$ (of which more later).

Functions of several variables also have total differentials. For $z = f(x,y)$ the total differential $dz$ is given by

$$dz = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy$$

As before, $dz$ approximates the change in $z$ for small increments $(dx, dy)$ in $x$ and $y$. This expression makes sense - if a surface is flat in the $x$-direction ($\partial f/\partial x=0$) then you can vary your $x$-position without affecting $z$.

**Error Analysis**

Let’s say that $z = f(x,y)$ and the uncertainties (errors) in $x$ and $y$ are $dx$ and $dy$, respectively. Then $dz$, the expected error in $z$, is given by

$$(dz)^2 = \left( \frac{\partial f}{\partial x} \, dx \right)^2 + \left( \frac{\partial f}{\partial y} \, dy \right)^2$$

where here we are squaring the terms (“adding in quadrature”) to avoid opposite signs cancelling the error. This approach assumes that the uncertainties are uncorrelated.

**Example** Say we want to estimate the shear modulus of a material from its density and shear velocity. If the values obtained for the density and shear velocity are $2700 \pm 40$ kg m$^{-3}$ and $3500 \pm 100$ m s$^{-1}$, respectively, what is the error in the resulting shear modulus? (Here we are using $v^2 = \mu/\rho$).
With a bit of algebra, you end up with

\[
\left(\frac{d\mu}{\mu}\right)^2 = \left(2\frac{dv}{v}\right)^2 + \left(\frac{d\rho}{\rho}\right)^2
\]

A good way of thinking about this is that \(d\mu/\mu\) is the fractional uncertainty in \(\mu\), which depends on the fractional uncertainty in \(v\) and \(\rho\). Note that the error is more sensitive to uncertainties in \(v\) than \(\rho\) because the relationship is that \(\mu = \rho v^2\).

Reminder: Problem Set 3 is due today by 6pm.