So far, all our calculus has been two-dimensional, involving only $x$ and $y$. Nature is three-dimensional, and things may also vary in time, so we need to extend calculus to higher dimensions.

3D Geometry

**When is it useful?** Anywhere you need to worry about three-dimensions . . .

We’ll start with 3D Cartesian space. This consists of three mutually perpendicular axes $x, y$ and $z$, ordered by the right hand rule.

![3D Cartesian space diagram]

**Figure 1**

A point is defined as an ordered triplet $(a, b, c)$ in this space. This is a short-hand way of writing three equations:

$$x = a$$
$$y = b$$
$$z = c$$

Just as in 2D we can calculate the distance between two points using Pythagoras, so we can in 3D. Here we are writing the distance between points $P_1$ and $P_2$ as $|P_1P_2|$.

![Distance between points diagram]

**Figure 2**
In 2D, we have

\[ |P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \]

In 3D, we have

\[ |P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \]

On a sphere, every point is equidistant from the centre of the sphere. So the equation of a sphere of radius \( r \) and centred at \((a, b, c)\) is

\[ r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2 \]

**What’s the 2D equivalent of this equation?**

**Vectors** A vector is a line segment in a particular direction: it has both magnitude (length) and an orientation. Scalars have magnitude, but no direction. E.g. speed is a scalar; velocity is a vector. A body travelling in a circle at constant speed has a continuously varying velocity.

In general, vectors are not fixed in space, but simply extend for a given distance in a given direction. A vector is denoted by \([a, b, c]\), which tells its extent in the \(x, y\) and \(z\) directions.

If two vectors are equal, then their \(x, y\) and \(z\) components must all be equal. Let \(v_1 = [a, b, c]\) and \(v_2 = [e, f, g]\). Then if \(v_1 = v_2\) we have

\[ a = e \]
\[ b = f \]
\[ c = g \]

Addition and subtraction of vectors is performed component by component

\[ v_1 + v_2 = [a + e, b + f, c + g] \]
\[ v_1 - v_2 = [a - e, b - f, c - g] \]

Graphically, we add vectors by placing the tail of \(v_2\) at the head of \(v_1\). **Note** that placing \(v_1\) at the head of \(v_2\) results in the same answer.
In subtraction, we reverse the direction of $\mathbf{v}_2$ ($-\mathbf{v}_2$ points in the opposite direction to $\mathbf{v}_2$). Note that $\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_2 - \mathbf{v}_1$ do not produce the same vector.

Reversing the direction of a vector is equivalent to multiplying it by -1. For any scalar number $h$ we have

$$h\mathbf{v} = h[a, b, c] = [ha, hb, hc]$$

The **magnitude** (length) of a vector is obtained from Pythagoras. Magnitude is a scalar, typically represented as $|\mathbf{v}|$. Numerically

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$$

It is therefore also true that

$$|h\mathbf{v}| = |h||\mathbf{v}| = h|\mathbf{v}|$$

**Rules**

1. Vector addition is commutative

   $$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

2. Vector addition is associative

   $$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

3. Scalar multiplication is distributive

   $$c(\mathbf{a} + \mathbf{b}) = ca + cb$$
There are three special types of vectors:

1. Null vector \( \mathbf{0} = [0, 0, 0] \). This vector has no length and no direction (and is not very useful).

2. Position vector \( \mathbf{p} = (a, b, c) \). Note that these are denoted by () not \([\ ]\). Position vectors have their tails at the origin and their head at the point \((a, b, c)\), so they are fixed in space, unlike other vectors. In a sense, position vectors just indicate points.

3. Unit vector. A unit vector has unit length: \( |\mathbf{v}| = 1 \). Any vector other than the null vector can be made into a unit vector as follows:

\[
\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}
\]

Here \( \hat{\mathbf{v}} \) denotes a unit vector (length=1). We often use unit vectors when only direction (and not magnitude) is important.

**Example** If \( \mathbf{v} = [3, 4, 5] \) then what is \( \hat{\mathbf{v}} \)?

**Unit Vector Notation**

In Cartesian co-ordinates, the unit vectors representing the \( x, y \) and \( z \) axes are particularly important. They are

\[
\hat{i} = [1, 0, 0] \\
\hat{j} = [0, 1, 0] \\
\hat{k} = [0, 0, 1]
\]

We can use \( \hat{i}, \hat{j}, \hat{k} \) to represent other vectors through scalar multiplication and vector addition. For example, if \( \mathbf{v} = [a, b, c] \) then we can rewrite this as

\[
\mathbf{v} = a\hat{i} + b\hat{j} + c\hat{k}
\]

In 2D, a unit vector pointing in an anti-clockwise direction \( \theta \) from the \( x \)-axis is \([\cos \theta, \sin \theta]\) (Why?).

**What about a vector 90° further around?**

**Spherical coordinates**

Since we live on a sphere, it is often useful to be able to relate position in terms of latitude \((\theta)\) and longitude \((\phi)\) to position in terms of \((x, y, z)\). By convention, the \( x \)-axis points along the direction of zero longitude (Greenwich meridian) and the \( z \)-axis points along the north pole.
If we are just interested in the unit position vector \((\hat{x}, \hat{y}, \hat{z})\) then you can show that (see Figure)

\[
\hat{x} = \cos \theta \cos \phi \\
\hat{y} = \cos \theta \sin \phi \\
\hat{z} = \sin \theta
\]

Note that \(\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 1\), as required for a unit vector.

You can also convert from \(\hat{x}, \hat{y}, \hat{z}\) back to lat,long by the following:

\[
\theta = \tan^{-1}\left(\frac{\hat{z}}{\sqrt{\hat{x}^2 + \hat{y}^2}}\right) = \sin^{-1} \hat{z} \\
\phi = \tan^{-1}\left(\frac{\hat{y}}{\hat{x}}\right)
\]

where \(\tan^{-1}\) means “inverse tan”. It is very useful to be able to convert from spherical to Cartesian and back, because some vector operations are much easier in Cartesian coordinates. We will see this in action next lecture.