Say we have a function \( f(x) \) which is complicated and we would like to simplify. Often, we can rewrite this equation in an approximate form \( f(x) \approx c_0 + c_1 x + c_2 x^2 + \cdots \) where \( c_1, c_2, \cdots \) are constants. This is incredibly useful when \( x \) is small (why?) - it is the single mathematical technique I (and many other people) use most frequently. We call this approach a series expansion of the particular function we are interested in.

**Example:** you have already been told that \( \cos \theta \approx 1 - \frac{\theta^2}{2} \) when \( \theta \) is small.

What we would like is a general way of determining the series expansion of a function \( f(x) \). There are a couple of ways of doing so. One way is graphical, and just depends on the definition of a gradient.

Say we know the value of a function \( f(x) \) at \( x=0: f(0) \). We want to know the value of this function at some other location \( x \). The gradient between these two points is just

\[
\frac{f(x) - f(0)}{x - 0}.
\]

As long as \( x \) is not too far away, then the gradient between these two points will be similar to \( f'(0) \), the value of the derivative at \( x = 0 \) (why?). So we can write

\[
f'(0) \approx \frac{f(x) - f(0)}{x}.
\]

Simple rearrangement then allows us to solve for \( f(x) \):

\[
f(x) \approx f(0) + xf'(0)
\] (1)

In other words, if we know the value of the function and its first derivative at \( x = 0 \), we can extrapolate to get the value of the function at other values of \( x \). This is very useful!

**Example.** Let’s say \( f(x) = \sin x \). Then we can use equation (1) to write

\[
\sin(x) \approx \sin(0) + x \cos(0) \approx x
\]

which is a good approximation as long as \( x \) is small (and in radians).
The accuracy of our extrapolation depends on how big $x$ is. To get a more accurate expression than equation (1), let’s just assume that the general expression is a polynomial:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$  \hspace{1cm} (2)

where $c_0, c_1, \cdots$ are the unknown constants that we want to find.

If we set $x = 0$ then we obtain $c_0 = f(0)$.

If we now differentiate our polynomial, we get

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

Again setting $x = 0$, we obtain $c_1 = f'(0)$. Substituting back into equation (2), we get

$$f(x) = f(0) + f'(0)x + c_2 x^2 + c_3 x^3 + \cdots$$

Ignoring the higher-order terms ($x^2, x^3$), we have now recovered equation (1).

But we can keep going. Repeating the procedure again and again, we get $2c_2 = f''(0)$, $3 \cdot 2c_3 = f'''(0)$, and so on. So we can determine all our unknown constants $c_0, c_1, \cdots$ in terms of the value of the function and its derivatives at $x = 0$.

Substituting these values back into our original polynomial, we end up with a series expansion for the function $f(x)$, known as Maclaurin’s Series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

Here $f^{(r)}$ denotes the $r$-th derivative of $f$ and $r! = r(r-1)(r-2)\cdots(2)(1)$ is called ”$r$-factorial”. Note that 0! = 1 (for reasons we won’t go into), and that $r!$ is not restricted to either positive or integer values of $r$ (though the expression for $r!$ given here only works for positive, integer values). Maclaurin’s series is a specific type of a more general class of series expansions, called Taylor series.

So what? Series expansions allow us to find simple approximations for complicated expressions. Example Let $f(x) = e^x$. Differentiating, we have

$$f^{(n)}(0) = 1$$

for all $n$. So using Maclaurin’s series we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
So if $x$ is small (such that $x^2 \ll x$) then we can approximate $e^x \approx 1 + x$ (try it on a calculator!).

What about $e^{2x}$?

**Example** Let $f(x) = (1 + x)^n$. If we differentiate once, we get

$$f'(x) = n(1 + x)^{n-1}.$$  

Setting $x=0$, we have $f'(0) = n(1 + 0)^{n-1} = n$.

Generalizing, we have

$$f^{(m)}(1 + x) = n(n-1)(n-2)\cdots(n-m+1)(1 + x)^{n-m}$$

and

$$f^{(m)}(0) = n(n-1)(n-2)\cdots(n-m+1)$$

Substituting all this back into the Maclaurin series, we get

$$(1 + x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots$$

This is called the *Binomial expansion* and is very useful. It works for positive, negative and fractional $n$. A particularly useful application is to remember that

$$(1 + x)^n \approx 1 + nx$$ when $x$ is sufficiently small.

For instance, if you had to evaluate $(1.1)^{10}$, you would rewrite this as $(1 + 0.1)^{10}$ and then approximate it using the Binomial expansion as $1 + (10 \times 0.1) = 2$, which is about 25% too small.

**How good is the approximation?**

This is a common question. The answer is that you can always look at the next term in the series expansion, and that will give you a good estimate of the error. So, continuing the example given above, the next term in the Binomial expansion is $(10)(9)/2 \times (0.1)^2 = 0.45$. So the error is approximately $0.45/2 \approx 25\%$.

**Example** We can similarly expand $\ln(1 + x)$. After a bit of algebra, we get

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$
**Example** The total relativistic energy of a particle $E$ is

$$E = mc^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

What does this reduce to when $v \ll c$?

**Example** What does

$$\frac{x - \sin x}{x^3}$$

reduce to as $x \to 0$?

**Hint** The main difficulty in carrying out a simplification of this kind is in recasting the expression into one in which one quantity is small. E.g. to find an approximate expression for $(20+3x)^5$, rewrite this as $20^5(1 + \frac{3x}{20})^5$ and then for $3x \ll 20$ you can approximate the answer using the first few terms of the binomial expansion $(1 + \frac{3x}{20})^5 = 1 + 5 \cdot \frac{3x}{20} + \cdots$.

First homework is due 6pm on Thursday!