Most of the sounds that you hear consist not of a single note (frequency), but a whole jumble of different frequencies. Your ear is very good at identifying a single frequency from all the background sound; in this lecture, we will show how any sound (or other signal) can be decomposed into a sum of sinusoidal waves of different frequencies. This is the topic of Fourier analysis.

Fourier analysis is absolutely central to any study of frequency-dependent phenomena, whether they be seismic waves, radio signals, stock market fluctuations, etc. etc. It allows you to solve more difficult nonhomogeneous higher-order differential equations. It also forms the basis for data compression and signal filtering.

In practice, Fourier analysis is often carried out numerically (using a piece of magic called the Fast Fourier Transform, or FFT). But it can also be done analytically, as the examples below will show.

Let’s start with the Maclaurin series:

\[ f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots = a_0 + a_1x + a_2x^2 + \cdots \]

This tells us that we can write down any function \( f(x) \) as a sum of a set of linearly independent functions: \( x, x^2, x^3 \cdots \).

For instance, we have

\[ \sin x = x - \frac{x^3}{3!} + \cdots \]

As you add higher and higher terms in \( x \), the approximation becomes better and better (try drawing this).

A Fourier series is very similar, except that now we are going to write down our function \( f(x) \) as a sum of a different set of linearly independent functions: \( \sin x, \sin 2x, \sin 3x \cdots \) (We can also add in \( \cos x, \cos 2x \cdots \) but I’m keeping it simple for now).

**Odd functions**

For a function which is ”odd” (anti-symmetric about the origin - like \( \sin x \)) we can write

\[ f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \] (1)

1
It turns out that any odd, periodic function can be represented by summing up a whole bunch of sine functions. This is a remarkable result.

All we need to do is to find $a_1, a_2$ etc. How do we do this?

Let’s assume that the function is defined between $-\pi$ and $\pi$. Then if we multiply both sides by $\sin mx$ and integrate, we get

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} a_1 \sin x \sin mx \, dx + \int_{-\pi}^{\pi} a_2 \sin 2x \sin mx \, dx + \int_{-\pi}^{\pi} a_3 \sin 3x \sin mx \, dx + \cdots$$

And here’s the trick: it turns out that $\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0$ unless $n = m$ (try it for $\sin x \sin 2x$ and $\sin x \sin 3x$).

So the above equation simplifies to

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} a_m \sin mx \sin mx \, dx$$

But we can show (how?) that

$$\int_{-\pi}^{\pi} a_m \sin mx \sin mx \, dx = a_m \pi$$

so that our solution for $a_m$ is given by

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx \quad (2)$$

So for any given function $f(x)$ we can find each coefficient $a_m$ using equation (2) and thus write down the Fourier series expansion for $f(x)$ using equation (1).

Here’s a worked example: find the Fourier series expansion of a square wave ($f(x) = -1$ for $-\pi < x < 0$, $f(x) = 1$ for $0 < x < \pi$).

We can split equation (2) into two domains, $-\pi < x < 0$ and $0 < x < \pi$. For the first domain, $f(x) = -1$ and for the second, $f(x) = 1$, so we have

$$a_m = \frac{1}{\pi} \int_{-\pi}^{0} (-1) \sin mx \, dx + \frac{1}{\pi} \int_{0}^{\pi} (1) \sin mx \, dx$$

which gives us $a_m = 4/\pi m$ for $m$ odd and $a_m = 0$ for $m$ even.

So making use of equation (1), our Fourier series expansion is

$$f(x) = \frac{4}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right) \quad (3)$$
In other words, we can make a square wave by summing up a whole bunch of sine waves. **Really?** The image below shows how equation (3) produces a better and better approximation to the square wave as you add in more and more higher-order terms.

![Figure 1](image_url)

**Even functions**

For a function which is "even" (symmetrical about the origin - like cos) we use a slightly different series expansion:

\[ f(x) = \frac{b_0}{2} + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \]

Note that the \(b_0\) term is divided by two. We can find the \(b_m\) in an exactly analogous way to equation (2):

\[ b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx \quad (4) \]

**Example** If we have a triangular function, \(f(x) = x\) for \(0 < x < \pi\) and \(f(x) = -x\) for \(-\pi < x < 0\), then we can show that

\[ f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left( \cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \cdots \right) \quad (5) \]

**Application to PDE’s**

Say we have found a general solution for the 2D steady-state heat conduction equation \(\nabla^2 T = 0\) (see Lecture 17) which gives us

\[ T = C \cos kx \, e^{-kz} \quad (6) \]

Let’s also assume that the boundary conditions at \(z = 0\) are given by a triangular function (equation 5). Recall that equation (6) is a solution for *any* \(k\), or combination of \(k\)’s. So you can
add up different solutions to satisfy the boundary conditions. The solution at $z = 0$ will therefore look something like

$$T = C_0 + C_1 \cos x + C_2 \cos 3x + C_3 \cos 5x + \cdots$$

where the $C$’s are constants given by equation (5). The full solution will be

$$T = C_0 + C_1 \cos x \, e^{-z} + C_2 \cos 3x \, e^{-3z} + C_3 \cos 5x \, e^{-5z} + \cdots$$

Note that each of the individual terms satisfies the governing PDE, but you need to series expansion to also match the boundary conditions. Also note that this equation implies that long wavelength signals (small $k$) penetrate to greatest depths.

**Mixed functions**

For a function which is neither odd nor even, you simply combine the two series expansions together:

$$f(x) = \frac{b_0}{2} + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots + a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \quad (7)$$

and find the $a_m$ and $b_m$ exactly as above.

**Why is Fourier Analysis useful?**

Each sine term and cosine term in the above expansions is associated with a particular wavelength. So Fourier series allow us to break any function down into a set of sines and cosines of different wavelengths. This is incredibly useful. Furthermore, we are not restricted to 1D - you can do Fourier transforms on 2D data too.

**Example** Say you have a signal (seismogram, radio, topographic profile, even a topo map) which is ”noisy”. If you know what the wavelength of the noise is (e.g. short-wavelength ”speckle”) you can filter out the noise as follows. 1) Find the Fourier series expansion. 2) Set the coefficients $(a_m, b_m)$ associated with the noise equal to zero. 3) Use the remaining coefficients to reconstruct the signal (using equation 7).

**Example** Attenuation of signals depends on their wavelength. To determine how a signal changes with distance, you break down the signal into its component wavelengths (using Fourier series expansion), work out how much each individual wavelength has been attenuated, and then recombine the attenuated wavelengths together using equation 7 to get the resulting signal.
Obviously, I am only scratching the surface here. The main point is that Fourier series allow you to break down any signal into its component wavelengths, and that turns out to be very useful.

**Reminders**

1. This Thursday will be a revision lecture - bring questions.
2. The Final is **Mon 11th Dec 4-7pm**. Several practice finals and the formula sheet will be posted on the website.
3. For those attending AGU, there will be an early final on Fri 8th Dec 9am-12 noon in D226.
4. Please fill out an online evaluation.