The last differential equation topic we are going to mention is higher-order PDE’s. These are almost invariably found in real physical problems, and we can only touch on them here. We will simply look at one technique which can simplify their solution; taking solutions further involves techniques such as Fourier analysis not covered in depth in this course.

**The Laplacian**

Recall how we describe heat conduction. In one dimension, heat conduction is governed by the equation

\[
\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}
\]

where \(\kappa\) is the thermal diffusivity and \(T(x,t)\) is a temperature field varying in \(x\) and time only.

In 2 dimensions, the analogous equation may be written

\[
\frac{\partial T}{\partial t} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) = \kappa \nabla^2 T
\]

where now \(T\) is a function of \(x, z\) and \(t\) and \(\nabla^2\) is called the Laplacian operator, also referred to as ”del-squared”. The Laplacian can be written for 2, 3 . . . dimensions; it has a slightly different explicit form in curvilinear coordinate systems.

The reason we use the Laplacian is that it is a useful shorthand, and that it works irrespective of which coordinate system we adopt. So \(\frac{\partial T}{\partial x} = \kappa \nabla^2 T\) is a true statement irrespective of whether we are using 2D, 3D, spherical, cylindrical etc. etc. Note, however, that the explicit form of the Laplacian does change depending on what coordinate system you’re in.

Another place where the Laplacian arises is in (not surprisingly) **Laplace’s equation**:

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f = 0
\]

Here \(f\) is a scalar quantity \(f(x,y,z)\) and this equation is written in Cartesian coordinates. This equation is fundamental to the description of gravity and charge, among other phenomena.

**Separation of Variables**
The problem in dealing with a PDE is that the function we are interested in depends on several variables (e.g. $T(x,t)$). One trick that we can try, called Separation of Variables, is to assume that the two variables are independent. That is, we can rewrite $T(x,t)$ as the product of a function that depends on $x$ only, and another that depends on $t$ only. This greatly simplifies matters.

**Example** Say we want to solve the heat diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

Then we rewrite $T(x,t) = A(x)B(t)$ and obtain

$$A \frac{dB}{dt} = \kappa B \frac{d^2 A}{dx^2}$$

**Note that** we have replaced partial derivatives with normal derivatives, because $A$ and $B$ are each functions of one variable only. We can divide through by $AB$ to obtain

$$\frac{1}{B} \frac{dB}{dt} = \frac{\kappa}{A} \frac{d^2 A}{dx^2}$$

Since the LHS is a function of $t$ only and the RHS is a function of $x$ only, then for this equality to be true, the expressions on both left and right must be constant (we’ll call it $c$). So we can write

$$\frac{1}{B} \frac{dB}{dt} = c \quad => \quad \frac{dB}{dt} - cB = 0$$

and

$$\frac{d^2 A}{dx^2} - \frac{c}{\kappa} A = 0$$

where $c$ is constant. **Note that** it is sometimes more useful to employ other constants, such as $-c$ or $c^2$, depending on the exact problem being dealt with (see below).

Now we have two OLDE’s, which we can solve as usual. For example, a solution for $B$ (remember that there are others) is

$$B = B_0 e^{ct}$$

where $B_0$ is another constant and we have assumed a boundary condition of $B = B_0$ at $t = 0$. Similarly, a solution to $A$ (again, there are others) is

$$A = A_0 e^{(\sqrt{c/\kappa})x}$$
and so one potential solution is that

\[ T = T_0 e^{c t} e^{(\sqrt{c/\kappa})x} \]

where we have folded the two constants \( A_0 \) and \( B_0 \) into a single constant \( T_0 \).

**Check: does this particular solution work?**

Choosing what to use for \( c \) is more of an art than a science, and is dictated by the physics of the problem and the boundary conditions. For instance, the algebra would have been simplified if I’d used \( c^2 \), rather than \( c \). If I’d used \(-c\) rather than \( c \) (which I’m allowed to do, since \( c \) is unknown), then the solution would have involved sines and cosines (that is, \( e^{icx} \) etc.) rather than exponentials. Why?

**Example** Using the same example, if the initial 1D temperature distribution is sinusoidal, how does it change over time?

[Answer: If we use \(-c\) rather than \( c \), then we end up with \( A = c_1 \sin \sqrt{c/\kappa}x + c_2 \cos \sqrt{c/\kappa}x \). And we also get \( B = B_0 \exp(-ct) \) so that the distribution decays exponentially with time.]

**Example** The 2D mantle flow caused by post-glacial rebound is described by the *biharmonic* equation

\[ \nabla^4 \psi = \frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \]

where \( \psi \) is a scalar *stream-function* which describes the velocity field and also ensures that the fluid is incompressible.

If we assume that the initial surface load is sinusoidal, then we can use separation of variables to write \( \psi = X(x)Y(y) \) and take \( X = \sin kx \), where \( k \) is the wavenumber.

This then gives us the constant coefficient HLDE:

\[ \frac{d^4 Y}{dy^4} - 2k^2 \frac{d^2 Y}{dy^2} + k^4 Y = 0 \]

which gives us four independent solutions for \( Y \) - **what are they?**

[Answer: The characteristic polynomial gives \((m^2 - k^2)^2 = 0\) which means that \( m = \pm k \) so we have repeated roots. The full answer is \( Y = c_1 e^{ky} + c_2 e^{-ky} + c_3 ye^{ky} + c_4 ye^{-ky} \).]

Separation of variables is useful in many physical situations, and need not be confined to 2-dimensions. For instance, if you solve Laplace’s equation in spherical (3D coordinates), then you
end up with *spherical harmonics*, which are the natural way of describing quantities like gravity or magnetic field on a sphere.

**Similarity Solutions**

This technique is almost opposite to separation of variables. Rather than assuming that the two variables can be treated independently, in a similarity solution you assume that they always appear together in some fixed ratio. This allows you to convert a PDE into an ODE (by going from two variables to one). For instance, thermal diffusion equations can be solved using a similarity variable $= x/\sqrt{t}$ (*why?*). Similarity solutions often require a good deal of intuition to be used successfully; further discussion of this topic is outside the scope of this class.

**Reminders**

1. Problem set is due next Monday, 5pm.